Subject M311 - LN01

Linear Algebra ~ theory and Methods to solve linear systems, is the most useful math tool, 75% of math problems in science & engineering involves solving a linear system at a stage.

What is a linear system? A system of linear equations.

A linear equation in n unknowns (variables) is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where the coefficients $a_1, \cdots, a_n$ and the right-hand side b are given real #'s, and $x_1, \cdots, x_n$ are n unknowns to be found.

A system of m linear equations in n unknowns is of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \text{An } \underbrace{m \times n}_{\text{size}} \text{ linear system}$$

where the coefficients a_{ij}'s and RHS b_i's are given real #'s, $x_1, \cdots, x_n$ are n unknowns to be found.

Ex: 2×2 system; 2×3 system 3×2 system

(a) $\begin{aligned} x_1+2x_2&=5 \\ 2x_1+3x_2&=8 \end{aligned}$; (b) $\begin{aligned} x_1-x_2+x_3&=2 \\ 2x_1+x_2-x_3&=4 \end{aligned}$; (c) $\begin{aligned} x_1+x_2&=2 \\ x_1-x_2&=1 \\ x_1&=4 \end{aligned}$.

Def: A solution to an $m\times n$ system is an ordered n-tuple $(x_1, \cdots, x_n)$ that satisfies all m equations.

Ex: $(x_1,x_2)=(1,2)$ is a solution to (a);
$(x_1,x_2,x_3)=(2,0,0)$ is a solution to (b). In fact, for any # α, $(x_1,x_2,x_3)=(2,\alpha,\alpha)$ is a solution to (b).
As for (c), the 3rd equation $x_1=4$ leads to $\begin{aligned} 4+x_2&=2 \\ 4-x_2&=1 \end{aligned}$

$\Rightarrow \begin{aligned} x_2&=-2 \\ x_2&=3 \end{aligned} \Rightarrow$ no solution.

* A linear system may have $\begin{cases} \text{solutions (consistent)} \begin{cases} \text{only one solution} \\ \text{many solutions.} \end{cases} \\ \text{no solution (Inconsistent)} \end{cases}$

(a) and (b) are consistent, (c) is inconsistent.

* The set of all solutions to a linear system is the solution set.

* By solving a linear system, we mean finding the solution set.

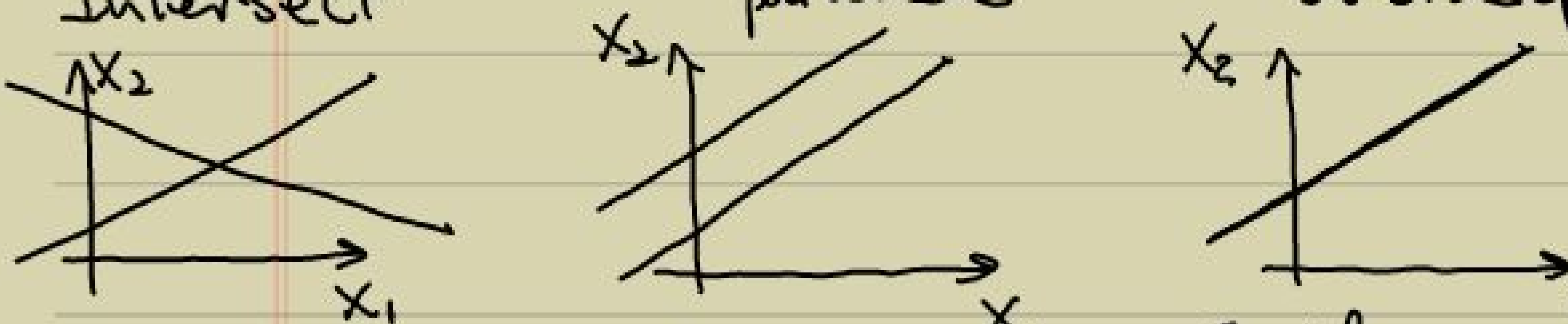
How to solve a linear system? How to find the solution set?

2x2 Systems: A 2x2 system is of the form

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2 \qquad (1)$$

In a 2D-plane, an equation $ax_1 + bx_2 = c$ represents a straight line. So a 2x2 system represents two lines in a 2D-plane. There are totally 3 cases:

Intersect | parallel | overlap



Consistent | Inconsistent | Infinitely many solutions

* A solution to a 2x2 system is the intersection point(s) of the two lines.

A linear equation in 3 variables $a_1x_1 + a_2x_2 + a_3x_3 = b$ represents a plane in a 3D-space.

Consider a 2x3 system: $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$.

Totally 3 cases:

1) Two planes intersect (along a line) consistent, infinitely many solutions.
2) Two planes are parallel (no solution) inconsistent;
3) Two planes are the same (consistent).

* An $m \times n$ system may have solution(s) (consistent) {only one / infinitely many}; no solution (inconsistent).

<u>Def</u>: Two systems are equivalent if they have exactly the same solution set.

Remark: Two equivalent systems must have the same # of variables.

* Three elementary operations that will not change the solution set:

1) Interchange the order of two equations;
2) Multiply a nonzero # to both sides of an equation;
3) Add a multiple of an equation to another equation.

* Two systems are equivalent if one system can be obtained by performing a sequence of three ele operations to another system.

* A system can be solved by first using those three ele operations and converting it into a simpler and equivalent system, then solve the simpler system.

Ex. (a) $\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$ (b) $\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 &= 3 \\ 2x_3 &= 4 \end{aligned}$

are equivalent. but (b) is much easier to solve $x_3 = 2$, $x_2 = 3$ $\Rightarrow x_1 = -2$

So to solve (a), we may convert (a) into (b) by using 3 ele operations: add (1) to (2) $\Rightarrow x_2 = 3$
$-1(1) + (3)$ $\Rightarrow 2x_3 = 4$ i.e, (b).

$n \times n$ systems have many special properties.

Def: An $n \times n$ system is in a triangular form if in the k-th equation, the first $k-1$ coefficients are all zero, i.e; $a_{ij} = 0$ if $i > j$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ 0 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\ \vdots \\ 0 + \cdots + 0 + a_{nn}x_n &= b_n \end{aligned}$$

$x_n \Rightarrow x_{n-1}, \cdots \Rightarrow x_2 \Rightarrow x_1$. back substitution.

Ex: $3x_1 + 2x_2 + x_3 = 1$
$x_2 - x_3 = 2$ is in triangular form
$2x_3 = 4$

Use back substitution: $x_3 = 2$, $\Rightarrow x_2 - 2 = 2 \Rightarrow x_2 = 4$, $\Rightarrow 3x_1 + 8 + 2 = 1$
$\Rightarrow x_1 = -3$. Thus $(x_1, x_2, x_3) = (-3, 4, 2)$ is the solution.

* Any $n \times n$ system can be converted into a triangular form by using 3 ele operations.

Ex. $x_1 + 2x_2 + x_3 = 3$ $\quad -3(1) + \to (2)$ $\quad x_1 + 2x_2 + x_3 = 3$

$3x_1 - x_2 - 3x_3 = -1$ $\quad -2(1) + \to (3)$ $\Rightarrow$ $\quad -7x_2 - 6x_3 = -10$

$2x_1 + 3x_2 + x_3 = 4$ $\quad\quad -x_2 - x_3 = -2$

$(2) \rightleftarrows (3) \Rightarrow$ $x_1 + 2x_2 + x_3 = 3$ $\quad -7(2) + \to (3)$ $\quad x_1 + 2x_2 + x_3 = 3$

$-x_2 - x_3 = -2$ $\quad \Rightarrow \quad -x_2 - x_3 = -2$

$-7x_2 - 6x_3 = -10$ $\quad\quad x_3 = 4.$

$x_3 = 4$, $-x_2 - 4 = -2 \Rightarrow x_2 = -2$. $x_1 + 2(-2) + 4 = 3 \Rightarrow x_1 = 3$

$(x_1, x_2, x_3) = (3, -2, 4)$ is the solution.