Subject §1.3 Matrix Algebra LN1.3.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})_{m\times n} \quad \begin{matrix} m\text{-rows} \\ n\text{-columns} \end{matrix}$$

a_{ij} – ijth entry $\begin{matrix} i\text{-th row} \\ j\text{-th column} \end{matrix}$

each row $(a_{i1}\ a_{i2} \cdots a_{in})$ is a $1\times n$ matrix, called a row vector.

each column $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is an $m\times 1$ matrix, called a column vector.

An n-tuple can be represent by either a $\begin{matrix}\text{row}\\ \text{column}\end{matrix}$ vector $\begin{matrix}[x_1, \cdots, x_n] \\ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\end{matrix}$.

samething, but different expression.

If $A_i^r = (a_{i1}\ a_{i2} \cdots a_{in})$, then $A = \begin{bmatrix} A_1^r \\ A_2^r \\ \vdots \\ A_m^r \end{bmatrix}$;

If $A_j^c = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, then $A = [A_1^c\ A_2^c \cdots A_n^c]$.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ columns $\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

rows $(1\ 2\ 3)$
$(4\ 5\ 6)$.

Matrix operations. Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$ $m\times n$ matrices.

Equality: $A = B \Leftrightarrow a_{ij} = b_{ij}$ for all $\begin{matrix} i = 1, \cdots, m \\ j = 1, \cdots, n. \end{matrix}$

Scalar multiplication: If α is a scalar, $\alpha A = (\alpha a_{ij})_{m\times n}$

Matrix Addition: $A \pm B = (a_{ij} \pm b_{ij})_{m\times n}$

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 11 & 8 & 5 \end{bmatrix}$

$$-2\begin{bmatrix} 1 & -2 & 3 \\ -6 & 5 & -4 \end{bmatrix} = \begin{bmatrix} -2 & 4 & -6 \\ 12 & -10 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 9 \\ 11 & 8 & 5 \end{bmatrix} - \begin{bmatrix} -2 & 4 & 6 \\ 5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 3 \\ 6 & 5 & 6 \end{bmatrix}$$

zero matrix $\theta = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$; $\theta + A = A + \theta = A$

$A + (-1)A = A - A = \theta.$

matrix multiplication: $A = (a_{ij})_{m\times n}$ $\quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$

$$Ax = \underset{m\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1}$$

Ex:

$$\underset{2\times 3}{\begin{bmatrix} 2 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 2+8+3 \\ 1+6+2 \end{bmatrix}} = \begin{bmatrix} 13 \\ 9 \end{bmatrix}_{2\times 1}$$

Now for an $m\times n$ system

$$(*) \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots & \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}; \quad A = \underset{m\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}, \; X = \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}, \; B = \underset{m\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

$(*) \Leftrightarrow Ax = B.$

If $A_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, $\cdots$; $A_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$,

then $(*) \Leftrightarrow x_1A_1 + x_2A_2 + \cdots + x_nA_n = B$.

$$2x_1 + 3x_2 - 2x_3 = 5$$

<u>Ex</u>: 2×3 system; $5x_1 - 4x_2 + 2x_3 = 6$

$$\Rightarrow \underset{2\times 3}{\begin{bmatrix} 2 & 3 & -2 \\ 5 & -4 & 2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 5 \\ 6 \end{bmatrix}} \quad \text{or} \quad x_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ -4 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

<u>Def</u>: If $v_1, v_2, \cdots, v_n$ are vectors in $\mathbb{R}^m$ and $c_1, c_2, \cdots, c_n$ are scalars, then a sum of the form

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

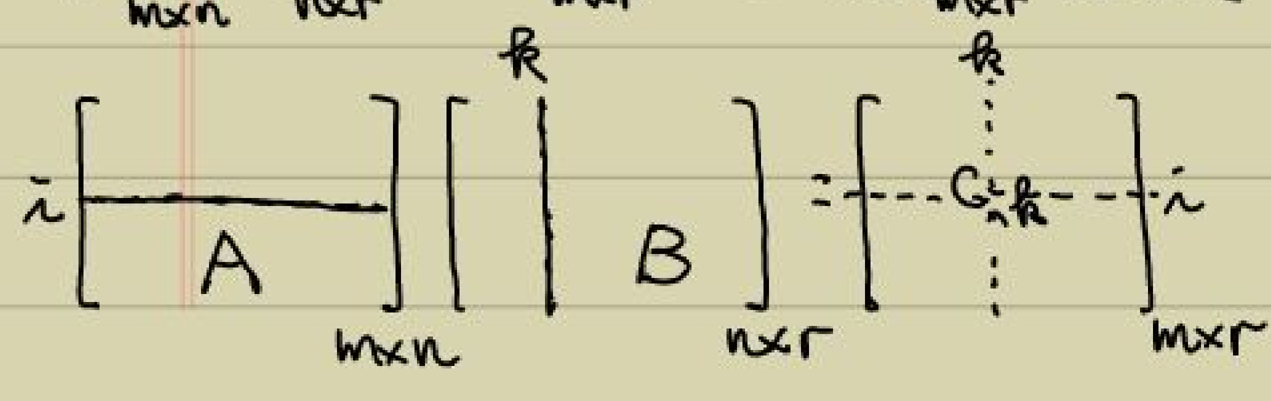
is called a <u>linear combination</u> of $v_1, v_2, \cdots, v_n$; $c_1, c_2, \cdots, c_n$ are the coefficients of the linear combination.

the above system is about to say: find coefficients x_1, x_2, x_3 such that the linear combination of $\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is equal to $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

* A linear system $Ax = b$ is consistent if and only if b can be written as a linear combination of the column vectors of A.

<u>multiplication</u>: $A = (a_{ij})_{m\times n}$ $B = (b_{jk})_{n\times r}$

$$A_{m\times n} B_{n\times r} = C_{m\times r} = (c_{ik})_{m\times r} \text{ where } c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$



Ex: $A=\begin{bmatrix}3&4\\1&2\end{bmatrix}_{2\times 2}$, $B=\begin{bmatrix}1&2\\4&5\\3&6\end{bmatrix}_{3\times 2}$

AB is not defined.

$$B_{3\times 2}A_{2\times 2}=\begin{bmatrix}1&2\\4&5\\3&6\end{bmatrix}_{3\times 2}\begin{bmatrix}3&4\\1&2\end{bmatrix}_{2\times 2}=\begin{bmatrix}5&8\\17&26\\15&24\end{bmatrix}_{3\times 2}$$

Properties of matrix operations

1) $A+B=B+A$,

2) $(A+B)+C=A+(B+C)$,

3) $(AB)C=A(BC)$,

4) $A(B+C)=AB+AC$,

5) $(A+B)C=AC+BC$,

6) $(\alpha\beta)A=\alpha(\beta A)$,

7) $\alpha(AB)=(\alpha A)B=A(\alpha B)$,

8) $(\alpha+\beta)A=\alpha A+\beta A$,

9) $\alpha(A+B)=\alpha A+\alpha B$.

* $A_{m\times n}B_{n\times r}$. AB is defined, BA is not defined if $m\neq r$.
Even when $m=r$, both AB and BA are defined
But $AB\neq BA$ in general.

Ex: $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$, $B=\begin{bmatrix}2&1\\-3&2\end{bmatrix}$.

$$AB=\begin{bmatrix}-4&5\\-6&11\end{bmatrix}\neq BA=\begin{bmatrix}5&8\\3&2\end{bmatrix}.$$

when $m=n$, $A_{n\times n}$ is called a square matrix. Then

Denote $$A^k = \underbrace{AA\cdots A}_{k\text{ copies}}.$$

Ex: $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$. $A^2=\begin{bmatrix}2&2\\2&2\end{bmatrix}$, $A^3=\begin{bmatrix}4&4\\4&4\end{bmatrix}$, $\cdots$, $A^k=\begin{bmatrix}2^{k-1}&2^{k-1}\\2^{k-1}&2^{k-1}\end{bmatrix}$.

the identity matrix $I=\begin{bmatrix}1&0&\cdots&0\\0&1&0\cdots&0\\ \cdots&&&\\0&\cdots&0&1\end{bmatrix}_{n\times n}$. $A_{m\times n}I_{n\times n}=A_{m\times n}$, $I_{m\times m}A_{m\times n}=A_{m\times n}$.

Ex: $A=\begin{bmatrix}2&3&4\\5&6&8\\7&1&9\end{bmatrix}_{3\times 3}$. then $AI=IA=A$.

Define $e_1=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$, $e_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$, $\cdots$, $e_n=\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$, then $I_{n\times n}=[e_1\ e_2\cdots e_n]$.

Transpose $(A^T)_{n\times m}$ of a matrix $A_{m\times n}$.

Columns of A^T are rows of A,

rows of A^T are columns of A.

Ex: $A=\begin{bmatrix}1&3&5\\2&4&6\end{bmatrix}_{2\times 3}$. $A^T=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}_{3\times 2}$.

$B=\begin{bmatrix}1&3&5\\2&4&6\\-1&-3&-5\end{bmatrix}_{3\times 3}$, $B^T=\begin{bmatrix}1&2&-1\\3&4&-3\\5&6&-5\end{bmatrix}_{3\times 3}$.

Def: An $n\times n$ matrix A is symmetric if $A^T=A$.

Ex: $A=\begin{bmatrix}1&3&5\\3&4&6\\5&6&2\end{bmatrix}$. $A^T=A$.

Properties

1) $(A^T)^T=A$,

2) $(\alpha A)^T=\alpha A^T$,

3) $(A+B)^T=A^T+B^T$,

* 4). $(A_{m\times n}B_{n\times r})^T_{r\times m}=(B^T)_{r\times n}\cdot(A^T)_{n\times m}$ a $r\times m$ matrix.