Subject §1.4 Elementary Matrices and Inverse

Def: Let A be an $n\times n$ matrix. If there is an $n\times n$ matrix B such that
$$AB = BA = I,$$
then A is called invertible (nonsingular) and $A^{-1} = B$ is called the inverse of A.
Otherwise A is said to be noninvertible (singular).

Ex: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If $d = a_{11}a_{22} - a_{12}a_{21} \neq 0$, then
$$A^{-1} = \frac{1}{d}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

check:
$$AA^{-1} = \frac{1}{d}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{d}\begin{bmatrix} a_{11}a_{22} - a_{12}a_{21}, & -a_{11}a_{12} + a_{12}a_{11} \\ a_{21}a_{22} - a_{22}a_{21}, & -a_{21}a_{12} + a_{22}a_{11} \end{bmatrix}$$
$$= \frac{1}{d}\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Ex: $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$. $d = 2 - 12 = -10 \neq 0$ $A^{-1} = \frac{-1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$.

* If $A_{n\times n}$ is invertible, then A^{-1} is unique.

suppose there is B such that $BA = AB = I = A^{-1}A = AA^{-1}$.
$BA = A^{-1}A$ multiply both sides by A^{-1} from the right $\Rightarrow B = A^{-1}$.

THM: If $A_{n\times n}$, $B_{n\times n}$ are invertible, then so is AB and
$$(AB)^{-1} = B^{-1}A^{-1}$$

$B^{-1}A^{-1}AB = B^{-1}B = I$, $\quad AB\,B^{-1}A^{-1} = AA^{-1} = I$. By the uniqueness.

§1.5. Elementary Matrices. $E_{\mathrm{I}}, E_{\mathrm{II}}, E_{\mathrm{III}}$

Apply 3 ele row operations to $A = (a_{ij})_{m \times n}$.

1) Interchange ith row with jth row of $A \Leftrightarrow E_{\mathrm{I}} A$;

2). multiply ith row of A by $\alpha \neq 0 \Leftrightarrow E_{\mathrm{II}} A$;

3) Add a multiply of ith row to jth row of $A \Leftrightarrow E_{\mathrm{III}} A$,

where 3 ele matrices are

1) E_{I} = Interchange ith row with jth row of I.

$$E_{\mathrm{I}} = \begin{array}{c c} \begin{bmatrix} 1 & 0 & \cdots & \vdots & \cdots & 0 \\ \cdots & 0 & \cdots & 1 & \cdots & \\ \cdots & 1 & \cdots & 0 & \cdots & \\ & \vdots & & \vdots & & \end{bmatrix} & \begin{array}{c} \\ i \\ j \\ \\ \end{array} \\ \begin{array}{cccccc} & i & & j & & \end{array} & \end{array};$$

2). $$E_{\mathrm{II}} = \begin{array}{c c} \begin{bmatrix} 1 & & & & O \\ & \ddots & & & \\ & & \alpha & & \\ & & & 1 & \\ O & & & & \ddots\ 1 \end{bmatrix} & \begin{array}{c} \\ \\ i \\ \\ \\ \end{array} \\ \begin{array}{ccccc} & & i & & \end{array} & \end{array} = \text{multiply } i\text{th row of } I \text{ by } \alpha \neq 0;$$

3). E_{III} = add a multiple α of ith row to jth row of I

$$E_{\mathrm{III}} = \underset{i<j}{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & \underset{ji}{\alpha} & & 1 \end{bmatrix}} \quad \underset{i>j}{\begin{bmatrix} 1 & & \alpha_{ji} & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}$$

: A α to jith entry in I.

Ex: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

1) Interchange 2nd and 3rd rows.

$$E_{\mathrm{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_{\mathrm{I}} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

2) Multiply the 2nd row by 2. $E_{II} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$E_{II}A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

3) $3(2) \to (1)$, $E_{III} = \begin{bmatrix} 1 & 3_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $E_{III}A = \begin{bmatrix} 3a_{21}+a_{11} & 3a_{22}+a_{12} & 3a_{23}+a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$3(1) \to (2)$, $E_{III} = \begin{bmatrix} 1 & 0 & 0 \\ 3_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{III}A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{11}+a_{21} & 3a_{12}+a_{22} & 3a_{13}+a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

<u>THM.</u> If E is an ele matrix, then E is nonsingular and E^{-1} is the same type. (think about the physical meaning of inverse: (to undo an operation))

<u>Proof</u> $E_I = \begin{array}{c} \\ \begin{bmatrix} 1 & & & & \\ & 0 & \cdots & 1 & \\ & \vdots & & \vdots & \\ & 1 & \cdots & 0 & \\ & & & & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ i \\ \\ j \\ \\ \end{array}$; $E_I^{-1} = E_I$ (to undo E_I).

$E_{II} = \begin{bmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & \alpha & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}$, $E_{II}^{-1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & 1/\alpha & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}$ (to undo E_{II})

E_{III} = add α to the jith entry in I. (to undo it)

E_{III}^{-1} = add $-\alpha$ to the jith entry in I.

<u>Ex:</u> $E_{III} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $E_{III}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$.

Def: A matrix B is row equivalent to A if

$$B = E_1 E_2 \cdots E_n A,$$

where $E_1, E_2, \cdots, E_n$ are ele matrices.

* If B is row equivalent to A, then A is row equivalent to B.

$B = E_1 E_2 \cdots E_n A \Leftrightarrow A = E_n^{-1} \cdots E_2^{-1} E_1^{-1} B.$

* If A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

$$A = E_1 E_2 \cdots E_n B,\ B = \overline{E}_1 \overline{E}_2 \cdots \overline{E}_m C,\ \Rightarrow A = E_1 \cdots E_n \overline{E}_1 \cdots \overline{E}_m C.$$

*THM. For $A_{n\times n}$, the following statements are equivalent.

1). A is nonsingular;

2). $Ax = 0$ has only the zero solution $x = 0$,

3). A is row equivalent to I,

4). $Ax = b$ has a unique solution for each b in $\mathbb{R}^n$.

Proof. 1) $\Rightarrow$ 2). $A^{-1}Ax = x = 0$.

2) $\Rightarrow$ 3). Row reduced Echelon form of $A = I$.

If one entry on the main diagonal $= 0 \Rightarrow$ multiple solutions.

3) $\Rightarrow$ 4). $E_n E_{n-1} \cdots E_2 E_1 A = I \Rightarrow \underbrace{x = E_n \cdots E_2 E_1 b}_{\text{unique}}$.

3) $\Rightarrow$ 1). $E_n \cdots E_2 E_1 A = I \Rightarrow A^{-1} = E_n \cdots E_2 E_1$.

4) $\Leftrightarrow$ 2). 4) $\Rightarrow$ 2). For 2) $\Rightarrow$ 4)

If $Ax_1 = b$, $Ax_2 = b$, then $A(x_1 - x_2) = b - b = 0$.

By 2). $x_1 - x_2 = 0 \Leftrightarrow x_1 = x_2 \Rightarrow$ 4).

4) $\Rightarrow$ 1). $Ax_1 = e_1$, $Ax_2 = e_2$, ---; $Ax_n = e_n$

$\Rightarrow A[x_1\, x_2 \cdots x_n] = [e_1\, e_2 \cdots e_n] = I. \Rightarrow A^{-1} = [x_1 x_2 \cdots x_n]$.

$\Rightarrow$ 1).

$*$ $E_n \cdots E_2 E_1 A = I \Rightarrow A^{-1} = E_n \cdots E_2 E_1$.

Ex: Find A^{-1} if

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

Put $[A|I]$. Apply 3 ele operations to reduce A to I.
and apply the same operations to I.

$$[A|I], [E_1 A | E_1 I], \cdots [\underbrace{E_n \cdots E_2 E_1 A}_{I} | \underbrace{E_n \cdots E_2 E_1 I}_{A^{-1}}]$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2(1)+\to(3)]{(1)+\to(2)} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{3(2)+\to(3)} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/6 & 1/2 & 1/6 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1/2 & -3/2 & -1/2 \\ 0 & 2 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1/6 & 1/2 & 1/6 \end{array}\right] \xrightarrow{-2(2)+\to(1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 2 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1/6 & 1/2 & 1/6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/4 & -1/4 & -1/4 \\ 0 & 0 & 1 & 1/6 & 1/2 & 1/6 \end{array}\right].$$

$\quad\quad I \quad\quad\quad\quad A^{-1}$

Diagonal matrix $\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}$ $a_{ij} = 0$ $i \neq j$

Ex: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Upper triangular Matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$ $a_{ij} = 0$ $i > j$

lower triangular matrix $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 \cdots & 0 \\ \cdots & & \ddots & 0 \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$ $a_{ij} = 0$ $i < j$