M311-10.3

Tuesday, April 4, 2023 1:50 AM

$F = Mi + Nj$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow F = (M, N) = \nabla f = (f_x, f_y)$

use two equations $\begin{cases} M = f_x \\ N = f_y \end{cases}$ to find f

$F = Mi + Nj + Pk$

$\nabla \times F = 0 \Rightarrow F = \nabla f \Rightarrow (M, N, P) = (f_x, f_y, f_z)$

three equations from which we can find f.

Use comparing and matching to find f

not adding same terms.

#6 $F = \frac{xy^2}{(1+x^2)^2} i + \frac{x^2 y}{1+x^2} j$

$$\frac{\partial M}{\partial y} = \frac{2xy}{(1+x^2)^2}, \quad \frac{\partial N}{\partial x} = \frac{2xy}{1+x^2} - \frac{x^2 y(2x)}{(1+x^2)^2} = \frac{2xy(1+x^2) - 2x^3 y}{(1+x^2)^2}$$

$$= \frac{2xy}{(1+x^2)^2} = \frac{\partial M}{\partial y} \Rightarrow \text{conservative}$$

To find f from $F = (M, N) = \nabla f = (f_x, f_y)$ i.e. $M = f_x$, $N = f_y$

$$f = \int M dx + \alpha(y) = \int \frac{xy^2}{(1+x^2)^2} dx + \alpha(y)$$

$$= \int N dy + \beta(x) = \int \frac{x^2 y}{1+x^2} dy + \beta(x) = \frac{1}{2} \frac{x^2 y^2}{1+x^2} + \beta(x)$$

Use $f_x = M$. $\Rightarrow$

$$\frac{xy^2}{(1+x^2)^2} = \frac{y^2}{2}\left(\frac{2x}{1+x^2} - \frac{x^2 \cdot 2x}{(1+x^2)^2}\right) + \beta'(x) = \frac{y^2}{2}\left(\frac{2x(1+x^2)-2x^3}{(1+x^2)^2}\right) + \beta'(x)$$

$$= \frac{y^2}{2}\frac{2x}{(1+x^2)^2} + \beta'(x) \Rightarrow \beta'(x) = 0 \text{ choose } \beta(x) = 0$$

thus

$$f(x,y) = \frac{1}{2}\frac{x^2y^2}{1+x^2}.$$

Check $f_y = N = \frac{x^2 y}{1+x^2}$, $f_x = M = \frac{xy^2}{(1+x^2)^2}$.

symmetric in x and y.

#5 $F = \left(3x^2 \cos y + \frac{y}{1+x^2y^2},\ x^3 \sin y + \frac{x}{1+x^2y^2}\right)$

$$\frac{\partial M}{\partial y} = -3x^2 \sin y + \frac{1}{1+x^2y^2} - \frac{y \cdot 2x^2 y}{(1+x^2y^2)^2}$$

$$= -3x^2 \sin y + \frac{1+x^2y^2-2x^2y^2}{(1+x^2y^2)^2}$$

$$= -3x^2 \sin y + \frac{1-x^2y^2}{(1+x^2y^2)^2}$$

$$\frac{\partial N}{\partial x} = 3x^2 \sin y + \frac{1-x^2y^2}{(1+x^2y^2)^2} \neq \frac{\partial M}{\partial y}.$$ not conservative

If $F = \left(-3x^2 \cos y + \frac{y}{1+x^2y^2},\ x^3 \sin y + \frac{x}{1+x^2y^2}\right)$

then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ conservative.

To find f from $M = f_x$ and $N = f_y$ $\tan^{-1}(x)$

#13 $F = (2x+y)\,i + (z\cos yz + x)\,j + (y\cos yz)\,k$

$$\nabla \times F = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2x+y & z\cos yz + x & y\cos yz & 2x+y & z\cos yz + x \end{array}\right|$$

$= i(\cos yz - yz\sin yz) + 0j + k$

$- k - i(\cos yz - yz\sin yz) - 0j = 0 \Rightarrow$ Conservative

$f = \int M\,dx + \alpha(y,z) = \int (2x+y)\,dx + \alpha(y,z) = x^2 + xy + \alpha(y,z)$

$= \int N\,dy + \beta(x,z) = \int (z\cos yz + x)\,dy + \beta(x,z) = \sin yz + xy + \beta(x,z)$

$= \int P\,dz + \gamma(x,y) = \int y\cos yz\,dz + \gamma(x,y) = \sin yz + \gamma(x,y)$

Compare and Match: $\Rightarrow \alpha(y,z) = \sin yz,\ \beta(x,z) = 0$ x^2, $\gamma(x,y) =$ x^2+ xy

$\Rightarrow f(x,y,z) = x^2 + xy + \sin yz$