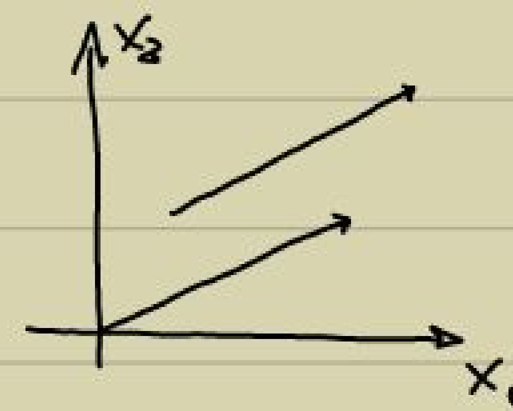
Subject §3.1. Vector spaces.

In $\mathbb{R}^2$, any nonzero vector can be represented by a directed line segment (by its direction and length). If we identify all the line segments with same direction and length, then vectors from $(0,0)$ to (x_1,x_2) and from (a_1,a_2) to (a_1+x_1, a_2+x_2)



are the same. We use (x_1,x_2) as the representative of its equivalent class. The length is

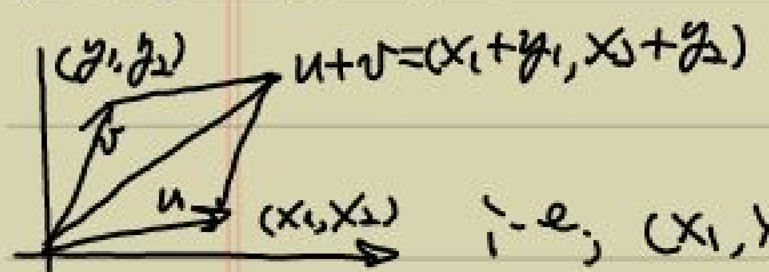
$$|(x_1,x_2)| = \sqrt{x_1^2+x_2^2}.$$

scalar multiplication:

$$\alpha(x_1,x_2) = (\alpha x_1, \alpha x_2) \begin{cases} \alpha>0, & \text{same direction, different length;} \\ \alpha<0, & \text{opposite direction different length;} \end{cases}$$

$|\alpha|>1$, enlarge the length.
$|\alpha|<1$, shrink the length.

vector addition



i.e., $(x_1,x_2)+(y_1,y_2) = (x_1+y_1, x_2+y_2)$.

similarly in $\mathbb{R}^n$, $u=(x_1,\cdots,x_n)$, $v=(y_1,\cdots,y_n)$ $\begin{matrix}\text{row}\\\text{column}\end{matrix}$ vectors

scalar multiplication: $\alpha u=(\alpha x_1,\alpha x_2,\cdots,\alpha x_n)$.

vector addition: $u+v=(x_1+y_1,x_2+y_2,\cdots,x_n+y_n)$.

In $\mathbb{R}^{m\times n}$ = set of all $m\times n$ matrices:

$$A=(a_{ij})_{m\times n} \quad B=(b_{ij})_{m\times n}.$$

scalar multiplication: $\alpha A=(\alpha a_{ij})_{m\times n}$.

vector addition: $A+B=(a_{ij}+b_{ij})_{m\times n}$.

Abstract vector space.

Def. A collection V of elements (called vectors) is a vector space if there are two operations '$\cdot$' = scalar multiplication and '$+$' = vector addition defined on V satisfying (Axioms)

A_1: $x+y=y+x$, any x,y in V;

A_2, $(x+y)+z=x+(y+z)$, any x,y,z in V;

A_3: there is a vector θ in V such that $x+\theta=\theta+x=x$ for any x in V;

A_4; For each x in V, there is $-x$ in V s.t. $x+(-x)=\theta$;

A_5: $\alpha(x+y)=\alpha x+\alpha y$, any α in $\mathbb{R}$, x,y in V;

A_6: $(\alpha+\beta)x=\alpha x+\beta x$, any α,β in $\mathbb{R}$, x in V;

A_7: $(\alpha\beta)x = \alpha(\beta x)$, any α, β in $\mathbb{R}$, x in V;

A_8: $1 \cdot x = x$, any x in V.

* vector space $(V, \text{`}\cdot\text{'}, \text{`}+\text{'})$.

Ex, $\mathbb{R}^{m\times n}$ with scalar multiplication "$\cdot$" and matrix addition "$+$" forms a vector space.

Ex: The vector space $C[a,b]$ = the set of all real-valued continuous functions on $[a,b]$.

Define "$\cdot$" $(\alpha f)(x) = \alpha f(x)$ for any α in $\mathbb{R}$, f in $C[a,b]$.

"$+$" $(f+g)(x) = f(x) + g(x)$, f, g in $C[a,b]$.

check $A_1, \cdots, A_8 \Rightarrow C[a,b]$ is a vector space.

Ex: P_n = the set of all polynomials of degree $\leq n$.

define "$\cdot$" and "$+$" by

$$(\alpha p)(x) = \alpha p(x), \quad \alpha p \text{ in } P_n$$

$$(p+q)(x) = p(x) + q(x), \quad p+q \text{ in } P_n.$$

verify $A_1, \cdots, A_8 \Rightarrow P_n$ is a vector space.

<u>THM</u>: If $(V, " \cdot ", " + ")$ is a vector space, then

1) $(-1)x = -x$, for any x in V,

2) $x + y = \theta \Leftrightarrow x = -y$,

3) $0 \cdot x = \theta$ for any x in V.

§3.2.

<u>Def</u>: Let $(V, " \cdot ", " + ")$ be a vector space and S be a subset of V. If S is closed under "$\cdot$" and "$+$", i.e.,

1) $\alpha x \in S$, $\forall \alpha \in \mathbb{R}$ and $x \in S$;

2) $x + y \in S$, $\forall x, y \in S$,

then S is called a subspace of V.

<u>Remarks</u>: 1) θ and V are subspaces of $(V, " \cdot ", " + ")$;

2) $(S, " \cdot ", " + ")$ is a vector space itself;

3). $\theta \in S$ a subspace;

4). A subset is described by a collection of elements in V satisfying certain condition(s). Check

a) αx satisfies the condition(s) if x does,

b) $x + y$ satisfies the condition(s) if x and y do.

If yes, then $\Rightarrow$ subspace, otherwise not.

$\underline{Ex.}$ $V=\mathbb{R}^2$, $S=\{(x_1,x_2): x_2=3x_1\} = \{(x_1,3x_1)\}$ is a subspace.

$$\left.\begin{array}{l} 1)\ \alpha\in\mathbb{R},\ x=(x_1,x_2)\in S \Rightarrow x=(x_1,3x_1),\ \alpha x=(\alpha x_1, 3\alpha x_1)\in S \\ 2)\ x=(x_1,3x_1),\ y=(y_1,3y_1)\in S \Rightarrow x+y=(x_1+y_1, 3(x_1+y_1))\in S \end{array}\right\}\text{yes.}$$

$\underline{Ex.}$ $V=\mathbb{R}^2$, $S=\{(x_1,x_2): x_1=1\}$ is not a subspace.

simply $\theta\notin S$. $\alpha x\in S$ if $x\in S$ and $\alpha\neq 1$.

$x+y\notin S$ if $x,y\in S$. $(1+1=2)$.

$\underline{Ex.}$ $V=\mathbb{R}^{2\times 2}$. $S=\left\{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}: a_{12}=-a_{21}\right\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

$$=\left\{\begin{bmatrix} a_{11} & a_{12} \\ -a_{12} & a_{22}\end{bmatrix}\right\}.$$

$$\left.\begin{array}{l} 1)\ A\in S.\ \alpha A=\begin{bmatrix}\alpha a_{11} & \alpha a_{12} \\ -\alpha a_{12} & \alpha a_{22}\end{bmatrix}\in S; \\ A,B\in S,\ A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ -a_{12}-b_{12} & a_{22}+b_{22}\end{bmatrix}=\begin{bmatrix} c_{11} & c_{12} \\ -c_{12} & c_{22}\end{bmatrix}\in S. \end{array}\right\}\text{yes.}$$

$\underline{Ex.}$ Let $C^m[a,b]$ be the set of all real functions f on $[a,b]$ with continuous mth derivative on $[a,b]$.

Then $C^n[a,b]$ is a subspace of $C^m[a,b]$ for all $n\geq m$.

$\underline{Ex.}$ Let $P_n[a,b]$ be the set of all polynomials of degree $\leq n$. then $P_n[a,b]$ is a subspace of $C^m[a,b]$ for any n.

Ex: Let $P_n^0[a,b]$ be the set of all polynomials of degree $\le n$ with $p(a)=0$. Then $P_n^0[a,b]$ is a subspace of $P_n[a,b]$.

Ex: For given continuous functions c_0, c_1, c_2.
Let S be the set of all f in $C^2[a,b]$ such that

$$c_2(x)f''(x)+c_1(x)f'(x)+c_0(x)f(x)=0.$$

Then S is a subspace of $C^2[a,b]$.

a) $f\in S$, $\quad c_2(x)(\alpha f)''(x)+c_1(x)(\alpha f)'(x)+c_0(x)(\alpha f)(x)$
$=\alpha\left(c_2(x)f''(x)+c_1(x)f'(x)+c_0(x)f(x)\right)=\alpha\cdot\theta=\theta$

b), $f,g\in S$, $\quad c_2(x)(f+g)''(x)+c_1(x)(f+g)'(x)+c_0(x)(f+g)(x)$
$=c_2(x)f''(x)+c_1(x)f'(x)+c_0(x)f(x)+c_2(x)g''(x)+c_1(x)g'(x)+c_0(x)g(x)$
$=\theta+\theta=\theta$

* Let A be an $m\times n$ matrix. Denote $N(A)=\{x\in\mathbb{R}^n : Ax=\theta\}$.
Then $N(A)$ is a subspace of $\mathbb{R}^n$ and called the nullspace of A.

$$\left.\begin{array}{l}\text{a). } x\in N(A),\ \alpha\in\mathbb{R}.\Rightarrow A(\alpha x)=\alpha Ax=\alpha\theta=\theta.\\ \text{b). } x_1,x_2\in N(A)\Rightarrow A(x_1+x_2)=Ax_1+Ax_2=\theta+\theta=\theta\end{array}\right\}\text{yes.}$$

Ex: Find $N(A)$ if $A=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

sol: Solve $Ax=0$ for all x. Augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-2(1)+(2)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

Freedom $=4-2=2$, set $x_2=s, x_3=t$.

$\Rightarrow x_1=-s-t,\ x_4=s+2t$

write $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$. any linear combination of the two vectors, a 2-d plane.

Def: Let $v_1, \cdots, v_n$ be in a vector space V. A sum of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some scalars $\alpha_1, \cdots, \alpha_n$, is called a linear combination of vectors $v_1, \cdots, v_n$. The set of all linear combinations of vectors $v_1, \cdots, v_n$ is called the span of $v_1, \cdots, v_n$, and denoted by $\text{Span}\{v_1, \cdots, v_n\}$.

theorem: $S=\text{span}\{v_1, \cdots, v_n\}$ is a subspace of V.

Proof: Since S is closed under operations "$\cdot$" and "$+$".

If $V_1=\text{span}\{v_1, \cdots, v_n\}$, we say $v_1, \cdots, v_n$ span V_1.

In the last example, we have $N(A) = \text{span}\left\{\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\2\end{bmatrix}\right\}$.

Ex: In $\mathbb{R}^n$, denote $e_1 = (1, 0, \cdots, 0)^T$, $e_2 = (0, 1, 0, \cdots, 0)^T$, $\cdots$

$e_n = (0, \cdots, 0, 1)^T$. Then $\mathbb{R}^n = \text{span}\{e_1, e_2, \cdots, e_n\}$, i.e.,

any vector $x = (x_1, \cdots, x_n)^T$ in $\mathbb{R}^n$ can be written as

linear combination of $\{e_1, \cdots, e_n\}$, indeed,

$$x = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = x_1\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} + \cdots + x_n\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix} = x_1e_1 + x_2e_2 + \cdots + x_ne_n.$$

Ex: Which of the following sets spans $\mathbb{R}^3$?

a) $\{e_1, e_2, e_3, (1,2,3)^T\}$;

b) $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$;

c) $\{(1,0,1)^T, (0,1,0)^T\}$;

d) $\{(1,2,4)^T, (2,1,3)^T, (4,-1,1)^T\}$.

Let $x = (x_1, x_2, x_3)^T$ be any vector in $\mathbb{R}^3$.

a) $x = x_1e_1 + x_2e_2 + x_3e_3 + 0(1,2,3)^T$. yes.

b) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \alpha_1\begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha_2\begin{bmatrix}1\\1\\0\end{bmatrix} + \alpha_3\begin{bmatrix}1\\0\\0\end{bmatrix}$ has a solution $(\alpha_1, \alpha_2, \alpha_3)$.

$$\Leftrightarrow \begin{array}{lll}\alpha_1 + \alpha_2 + \alpha_3 & = x_1 \\ \alpha_1 + \alpha_2 & = x_2 \\ \alpha_1 & = x_3\end{array} \uparrow \Rightarrow \begin{array}{l}\alpha_1 = x_3,\ \alpha_2 = x_2 - \alpha_1 = x_2 - x_3. \\ \alpha_3 = x_1 - \alpha_2 - \alpha_1 = x_1 - (x_2 - x_3) - x_3 = x_1 - x_2.\end{array}$$ yes

c) If $x_1 \neq x_3$, then there are no α_1, α_2 such that

$$\underbrace{\alpha_1\begin{bmatrix}1\\0\\1\end{bmatrix}+\alpha_2\begin{bmatrix}0\\1\\0\end{bmatrix}}_{=\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_1\end{bmatrix}}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}.$$

no.

d) Can we solve $\alpha_1\begin{bmatrix}1\\2\\4\end{bmatrix}+\alpha_2\begin{bmatrix}2\\1\\3\end{bmatrix}+\alpha_3\begin{bmatrix}4\\-1\\1\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ for $\alpha_1, \alpha_2, \alpha_3$

where $(x_1, x_2, x_3)^T$ is any vector in $\mathbb{R}^3$?

Equivalent statements.

1) $A_{n\times n}$ is nonsingular; 2) $Ax=0$ has only zero solution $x=0$.

3) A is row equivalent to I; 5). $|A| \neq 0$;

4) $Ax=b$ has a unique solution x for each b in $\mathbb{R}^n$.

So c) is to check 4), instead we check 5).

$$\begin{vmatrix}1 & 2 & 4\\2 & 1 & -1\\4 & 3 & 1\end{vmatrix} \overset{\substack{-2(1)+(2)\\-4(1)+(3)}}{=\!=\!=} \begin{vmatrix}1 & 2 & 4\\0 & -3 & -9\\0 & -5 & -15\end{vmatrix} = 0$$

} proportional

No.

<u>Ex</u>: Let $P_3 = \text{span}\{1, x, x^2\}$.

check $\text{span}\{1-x^2, x+2, x^2\} = P_3$.

a). Let $p = ax^2+bx+c$ be any vector in P_3.

can we find $\alpha_1, \alpha_2, \alpha_3$ such that

$$ax^2+bx+c = \alpha_1(1-x^2)+\alpha_2(x+2)+\alpha_3x^2$$

$$=(\alpha_3-\alpha_1)x^2+\alpha_2x+\alpha_1+2\alpha_2$$

By equating the coefficients of like power, we obtain a linear system

$$\begin{aligned} x^2 &: \ a = \alpha_3 - \alpha_1 \\ x &: \ b = \alpha_2 \\ 1 &: \ c = \alpha_1 + 2\alpha_2 \end{aligned} \iff \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0. \Rightarrow \text{yes.}$$

b). $1-x^2, x+2, x^2$ generate $1, x, x^2$ by liner combination

$$1 = (1-x^2) + x^2, \quad x = (x+2) - 2(1-x^2) - 2x^2, \quad x^2 = x^2.$$

Ex. Let $x_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$ in $\mathbb{R}^3$. $S = \text{span}\{x_1, x_2, x_3\}$.

Note $x_3 = 2x_2 + 3x_1$, $x_2 = -\frac{3}{2}x_1 + \frac{1}{2}x_3$, $x_1 = -\frac{2}{3}x_2 + \frac{1}{3}x_3$.

So $S = \text{span}\{x_1, x_2, x_3\} = \text{span}\{x_1, x_2\} = \text{span}\{x_1, x_3\}$

$= \text{span}\{x_2, x_3\}$.

can we further reduce the # of vectors to span S? No. $x_i \neq$ multiple of x_j. $i \neq j$.