Subject §3.4. Basis and Dimension

Def: The vectors $v_1, \cdots, v_n$ are said to form a basis for a vector space V if

1) $v_1, \cdots, v_n$ are LI (no more);

2) $V = \operatorname{span}\{v_1, \cdots, v_n\}$ (no less).

Remark: 1) Adding any vector to a basis will LD;

2) Taking away any vector away from a basis cannot span V.

Ex: Standard bases

1) $\mathbb{R}^n = \operatorname{span}\{e_1, \cdots, e_n\}$, $\quad e_i = (0, \cdots, 0, \overset{i\text{th}}{1}, 0 \cdots 0)$

2) $\mathbb{R}^{m\times n} = \operatorname{span}\{e_{ij} : {1 \le i \le m \atop 1 \le j \le n}\}$, $\quad e_{ij} = \begin{array}{cc} \left[\begin{array}{ccc} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{array}\right] & i \\ \quad j & \end{array}$

3) $P_{n+1} = \operatorname{span}\{1, x, \cdots, x^n\}$.

THM: If $V = \operatorname{span}\{v_1, \cdots, v_n\}$, then any collection of m vectors in V with $m > n$ is LD.

Proof: Let $u_1, \cdots, u_m$ be m vectors with $m > n$. Denote

$$u_i = a_{i1} v_1 + \cdots + a_{in} v_n = \sum_{j=1}^{n} a_{ij} v_j .$$

To show that $u_1, \cdots, u_m$ are LD, set

$$\sum_{i=1}^{m} c_i u_i = c_1 u_1 + \cdots + c_m u_m = 0$$

(show it has nonzero solutions for $c_1, \cdots, c_m$.)

$$= \sum_{i=1}^{m} c_i \sum_{j=1}^{n} a_{ij} v_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} c_i a_{ij} \right) v_j$$

Set $\sum_{i=1}^{m} c_i a_{ij} = 0 \quad j = 1, \cdots, n,$

$\Rightarrow$ n equations, m unknown $c_1, \cdots, c_m$.

Freedom $= m - n > 0$ $\Rightarrow$ there are nonzero solutions for $c_1, \cdots, c_m$. $\Rightarrow$ $u_1, \cdots, u_m$ are LD.

<u>Corollary</u> If $\{v_1, \cdots, v_n\}$ and $\{u_1, \cdots, u_m\}$ are two bases for a vector space V, then $m = n$.

<u>Def</u>: Let V be a vector space. If V has a basis consisting of n vectors, then we say that V has dimension n. $(\dim(V) = n)$

<u>Remark</u> 1) $\{0\}$ has dimension 0;

2) V is finite-dimensional if it has a finite basis. Otherwise V is infinite-dimensional.

THM: If V is a vector space with $\dim(V)=n$, then

1) any set of n LI vectors spans V;

2) any n vectors that span V are LI.

Proof. 1) Let $\{u_1, \cdots, u_n\}$ be a basis for V and $\{v_1, \cdots, v_n\}$ be LI. For any $v \in V$, $\{v_1, \cdots, v_n, v\}$ is LD, since $m=n+1>n$. $\Rightarrow c_1 v_1 + \cdots + c_n v_n + c_{n+1} v = 0$ has a nonzero solution. Since $v_1, \cdots, v_n$ are LI

$\Rightarrow c_{n+1} \neq 0 \Rightarrow v = \frac{-1}{c_{n+1}}(c_1 v_1 + \cdots + c_n v_n)$.

$\Rightarrow V = \text{span}\{v_1, \cdots, v_n\}$.

2). If $V = \text{span}\{v_1, \cdots, v_n\}$, but $\{v_1, \cdots, v_n\}$ is LD.

$\Rightarrow$ there is v_i s.t. $v_i = c_1 v_1 + \cdots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \cdots + c_n v_n$.

Thus $V = \text{span}\{v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n\}$

$\Rightarrow \dim(V) \leq n-1$. ※.

Ex: In $\mathbb{R}^n$, let $v_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, \cdots, v_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix}$.

How to determine if $\{v_1, \cdots, v_n\}$ is a basis for $\mathbb{R}^n$?

Put $\left| v_1\ v_2 \cdots v_n \right| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0 \Leftrightarrow \{v_1, \cdots, v_n\}$ is LI

$\Leftrightarrow \mathbb{R}^n = \text{span}\{v_1, \cdots, v_n\}$.

*Ex: In $\mathbb{R}^n$, let $v_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, \cdots, v_m = \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix}$.

How to check if $\{v_1, \cdots, v_m\}$ is LI ?

Put $[v_1 \cdots v_m] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \rightarrow$ [reduced row Echelon form] dependent relation among columns remain the same.

* 1) Columns with leading 1's are LI;
* 2) A column without a leading 1 is a linear combination of columns with leading 1's to the left.

Ex: Show that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^3$.

Note: dim = 3, # of vectors = 3. only have to show LI.

$$\left| \begin{matrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{matrix} \right| \begin{matrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{matrix} \Bigg| = \begin{matrix} 1+0+0 \\ -3-0+4 \end{matrix} = 2 \neq 0 \Rightarrow LI, \; 3 \sim 3 \Rightarrow \text{span}.$$

LI + span = basis.

THM. If V is a vector space with $\dim(V) = n > 0$, then

1) no set of less than n vectors can span V;
2) any set of less than n LI vectors can be extended to form a basis for V;
3) any set that spans V can be reduced to form a basis for V.

2) how to extend a LI set to form a basis?
3) how to reduce a spanning set to form a basis?

Ex. Extend $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ to form a basis for $\mathbb{R}^3$.

Using the standard basis e_1, e_2, e_3. Put

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 5 & -2 & 1 & 0 \\ 0 & 6 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{6}{5}} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 5 & -2 & 1 & 0 \\ 0 & 0 & -\frac{3}{5} & -\frac{6}{5} & 1 \end{bmatrix}$$

with leading 1's

$$\to \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & 2 & -\frac{5}{3} \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ LI, } 3 \sim 3 \Rightarrow \text{basis.}$$

Columns with leading 1' are LI.

Ex. Reduce $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ to form a basis for $\mathbb{R}^3$.

Put $\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 3 & 3 & 1 & 0 \\ 2 & 4 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$

with leading 1's LI

$3 \sim 3$. LI $\Rightarrow$ span. $\Rightarrow$ basis.

Def: Let V be a vector space with $\dim(V) = n > 0$ and $\mathcal{B} = \{v_1, \cdots, v_n\}$ be a basis. For each $v \in V$, v can be uniquely expressed as $v = c_1 v_1 + \cdots + c_n v_n$ for some scalars $c_1, \cdots, c_n$. $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ is called the coordinates of v w.r.t. the basis $\mathcal{B} = \{v_1, \cdots, v_n\}$.

standard bases.

$\mathbb{R}^n = \text{span}\{e_1, \cdots, e_n\}$. for any $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$.

$x = x_1 e_1 + \cdots + x_n e_n$. $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is the coordinates of x

w.r.t. the standard basis $\{e_1, \cdots, e_n\}$.

$P_{n+1} = \text{span}\{1, x, \cdots, x^n\}$. any $p(x) = a_n x^n + \cdots + a_1 x + a_0$

has the coordinate $(a_0, a_1, \cdots, a_n)^T \in \mathbb{R}^{n+1}$ w.r.t. the

standard basis $\{1, x, \cdots, x^n\}$. If the standard

basis $\{x^n, \cdots, x, 1\}$ is used for P_{n+1}, then the

coordinates of $p(x) = a_n x^n + \cdots + a_1 x + a_0$

is $(a_n, \cdots, a_1, a_0)^T \in \mathbb{R}^{n+1}$.