Subject §3.5 change bases.

Def. Let X be a vector space with $\dim(X)=n>0$
$V=\{v_1,\cdots,v_n\}$ be a basis for X.

For each $v\in X$, there is a unique expression

$$v=c_1v_1+\cdots+c_nv_n$$

for some scalar $c_1,\cdots,c_n$, called the coordinates of v w.r.t. the basis $V=\{v_1,\cdots,v_n\}$, denoted by

$$[v]_V=\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}\in\mathbb{R}^n.$$

$$* \quad [v]_V=\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}\Leftrightarrow v=c_1v_1+\cdots+c_nv_n.$$

standard bases: $\mathbb{R}^n=\text{span}\{e_1,\cdots,e_n\}$, $P_{n+1}=\text{span}\{1,x,\cdots,x^n\}$.

For the same v in X, when a basis is changed from V to U, the coordinates of v w.r.t. the basis V, $[v]_V$ is changed to the coordinates of v w.r.t. the basis U, $[v]_U$ correspondingly. Can we find a relation between the change in bases? Yes $[v]_V\sim[v]_U$, $V\sim U$.

Ex. Consider $x = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$ in $\mathbb{R}^3$. $\begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$ is actually the coordinates of x w.r.t. the standard basis $E = \{e_1, e_2, e_3\}$.

Let $\mathcal{V} = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ be another basis for $\mathbb{R}^3$.

Find the coordinates $[x]_{\mathcal{V}}$ of x w.r.t. the basis $\mathcal{V}$, i.e, solve $x = c_1 v_1 + c_2 v_2 + c_3 v_3$ for $[x]_{\mathcal{V}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Thus

$$\begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 5 & -2 & -7 \\ 0 & 6 & -2 & -8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 5 & -2 & -7 \\ 0 & 0 & 2/5 & 2/5 \end{array}\right] \Rightarrow \begin{matrix} c_3 = 1 \\ c_2 = -1 \\ c_1 = 2 \end{matrix}$$

Thus $[x]_{\mathcal{V}} = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}_{\mathcal{V}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Ex: Consider $P_3 = \text{span}\{1, x, x^2\}$. $p = x^2 - x + 3$.

If the standard basis $S = \{1, x, x^2\}$ is used, then

$[p]_S = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$. $\Leftrightarrow p = 3(1) - 1(x) + 1(x^2)$.

If the basis $\bar{S} = \{x^2, x, 1\}$ is used, then

$[p]_{\bar{S}} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \Leftrightarrow p = 1(x^2) - 1(x) + 3(1)$.

If the basis $U = \{x, x-1, x^2+1\}$ is used, then write

$x^2 - x + 3 = c_1 x + c_2(x-1) + c_3(x^2+1) = c_3 x^2 + (c_1 + c_2)x - c_2 + c_3$.

x^2: $1 = c_3$
x: $-1 = c_1 + c_2$ or $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \Rightarrow \begin{matrix} c_3 = 1 \\ c_1 = 1 \\ c_2 = -2 \end{matrix}$
1: $3 = -c_2 + c_3$

thus $[p]_U = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \Leftrightarrow 1(x) - 2(x-1) + 1(x^2+1) = x^2 - x + 3$.
check

Let $V = \{v_1, \cdots, v_n\}$ and $U = \{u_1, \cdots, u_n\}$ be two bases for X.

Let x be a vector in X.

$[x]_V = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \Leftrightarrow x = a_1 v_1 + \cdots + a_n v_n$.

$[x]_U = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Leftrightarrow x = b_1 u_1 + \cdots + b_n u_n$.

Thus $a_1 v_1 + \cdots + a_n v_n = b_1 u_1 + \cdots + b_n u_n$.

the transition matrix T from V to U is an $n \times n$ matrix such that $[x]_U = T[x]_V$ or $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = T \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ for each x in X.

Express vectors in V in terms of vectors in U:

$v_1 = t_{11} u_1 + \cdots + t_{n1} u_n$
$v_2 = t_{12} u_1 + \cdots + t_{n2} u_n$ $\Rightarrow T = \begin{bmatrix} \text{coefficients} \\ \text{in the expression} \end{bmatrix}^T = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ t_{21} & \cdots & t_{2n} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix}$.
$\cdots$
$v_n = t_{1n} u_1 + \cdots + t_{nn} u_n$

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when in $\mathbb{R}^n$, $a_1 v_1 + \cdots + a_n v_n = b_1 u_1 + \cdots + b_n u_n \Leftrightarrow$

$\begin{bmatrix} v_{11} & \cdots & v_{1n} \\ v_{21} & & v_{2n} \\ \vdots & \cdots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ u_{21} & \cdots & u_{2n} \\ \vdots & & \vdots \\ u_{n1} & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ or $V[x]_V = U[x]_U$

$\Leftrightarrow$

$$[x]_U = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ u_{21} & \cdots & u_{2n} \\ & \cdots & \\ u_{n1} & \cdots & u_{nn} \end{bmatrix}^{-1} \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ v_{21} & \cdots & v_{nn} \\ & \cdots & \\ v_{n1} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = T[x]_V \quad \text{i.e.,}$$

the transition matrix T from V to U is

$$T = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ u_{21} & \cdots & u_{2n} \\ & \cdots & \\ u_{n1} & \cdots & u_{nn} \end{bmatrix}^{-1} \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ v_{21} & \cdots & v_{2n} \\ & \cdots & \\ v_{n1} & \cdots & v_{nn} \end{bmatrix} \Leftrightarrow T = U^{-1}V,\ V = UT.$$

the transition matrix W from U to V is

$$W = T^{-1} = \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ & \cdots & \\ v_{n1} & \cdots & v_{nn} \end{bmatrix}^{-1} \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ & \cdots & \\ u_{n1} & \cdots & u_{nn} \end{bmatrix} = V^{-1}U.$$

In particular, if $U = \{e_1, \cdots, e_n\}$, then $T = \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ & \cdots & \\ v_{n1} & \cdots & v_{nn} \end{bmatrix} = V.$

if $V = \{e_1, \cdots, e_n\}$, then $T = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ & \cdots & \\ u_{n1} & \cdots & u_{nn} \end{bmatrix}^{-1} = U^{-1}.$

<u>Ex.</u> Find the transition matrix T from $\{v_1, v_2\}$ to $\{u_1, u_2\}$ where $v_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Put $V = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $U = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, we have $V[x]_V = U[x]_U$.

The problem asks for $[x]_U = T[x]_V$. Thus $T = U^{-1}V$.

$U^{-1} = \frac{1}{3-2}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$. $T = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}$.

If a vector v in $\mathbb{R}^2$ with $[v]_V = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$,

then $[v]_U = T[v]_V = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

2nd method:

Ask for $V \xrightarrow{T} U$. Express V in terms of U.

$v_1 = s_{11}u_1 + s_{21}u_2$, $\begin{bmatrix}5\\2\end{bmatrix} = s_{11}\begin{bmatrix}3\\2\end{bmatrix} + s_{21}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3&1\\2&1\end{bmatrix}\begin{bmatrix}s_{11}\\s_{21}\end{bmatrix}$.

$$\begin{bmatrix}s_{11}\\s_{21}\end{bmatrix} = \begin{bmatrix}3&1\\2&1\end{bmatrix}^{-1}\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}1&-1\\-2&3\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}3\\-4\end{bmatrix}.$$

$v_2 = s_{12}u_1 + s_{22}u_2$, $\begin{bmatrix}7\\3\end{bmatrix} = s_{12}\begin{bmatrix}3\\2\end{bmatrix} + s_{22}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3&1\\2&1\end{bmatrix}\begin{bmatrix}s_{12}\\s_{22}\end{bmatrix}$

$$\begin{bmatrix}s_{12}\\s_{22}\end{bmatrix} = \begin{bmatrix}3&1\\2&1\end{bmatrix}^{-1}\begin{bmatrix}7\\3\end{bmatrix} = \begin{bmatrix}1&-1\\-2&3\end{bmatrix}\begin{bmatrix}7\\3\end{bmatrix} = \begin{bmatrix}4\\-5\end{bmatrix}$$

$\Rightarrow T = \begin{bmatrix}3&4\\-4&-5\end{bmatrix}$. or $[v_1, v_2] = [u_1, u_2]\,T$

i.e, express of V in terms of U.

Ex: Consider $P_3 = \text{span}\{1, x, x^2\}$. Find the transition matrix T from the standard basis $\{1, x, x^2\}$ to a basis $\{1, 2x, 4x^2-2\}$

$V = \{v_1, v_2, v_3\}$

$U = \{u_1, u_2, u_3\}$

Ask for $V \xrightarrow{T} U$, need to express V in terms of U.

It is hard to find T directly

$1 = t_{11}1 + t_{21}(2x) + t_{31}(4x^2-2)$

$x = t_{12}1 + t_{22}(2x) + t_{32}(4x^2-2)$ $\qquad \Rightarrow T = \begin{bmatrix}1&0&1/2\\0&1/2&0\\0&0&1/4\end{bmatrix}$.

$x^2 = t_{13}1 + t_{23}(2x) + t_{33}(4x^2-2)$.

But it is easy to express U in terms of V.

$1 = 1$, $2x = 2\cdot x$, $4x^2-2 = -2(1) + 4x^2$.

the transition matrix τ from U to V is $\begin{bmatrix}1&0&-2\\0&2&0\\0&0&4\end{bmatrix}$.

thus $T = \tau^{-1} = \begin{bmatrix}1&0&-2\\0&2&0\\0&0&4\end{bmatrix}^{-1} = \begin{bmatrix}1&0&1/2\\0&1/2&0\\0&0&1/4\end{bmatrix}$.