SECTION 4 EXERCISES

1. Explain why each of the following algebraic rules will not work in general when the real numbers a and b are replaced by $n \times n$ matrices A and B.
   (a) $(a+b)^2 = a^2 + 2ab + b^2$
   (b) $(a+b)(a-b) = a^2 - b^2$

2. Will the rules in Exercise 1 work if a is replaced by an $n \times n$ matrix A and b is replaced by the $n \times n$ identity matrix I?

3. Find nonzero 2×2 matrices A and B such that $AB = O$.

4. Find nonzero matrices A, B, and C such that
$$AC = BC \quad \text{and} \quad A \neq B$$

5. The matrix
$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$
has the property that $A^2 = O$. Is it possible for a nonzero symmetric 2×2 matrix to have this property? Prove your answer.

6. Prove the associative law of multiplication for 2×2 matrices; that is, let
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$
$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$
and show that
$$(AB)C = A(BC)$$

7. Let
$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
Compute A^2 and A^3. What will A^n turn out to be?

8. Let
$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
Compute A^2 and A^3. What will A^{2n} and A^{2n+1} turn out to be?

9. Let
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
Show that $A^n = O$ for $n \geq 4$.

10. Let A and B be symmetric $n \times n$ matrices. For each of the following, determine whether the given matrix must be symmetric or could be nonsymmetric:
    (a) $C = A + B$ (b) $D = A^2$
    (c) $E = AB$ (d) $F = ABA$
    (e) $G = AB + BA$ (f) $H = AB - BA$

11. Let C be a nonsymmetric $n \times n$ matrix. For each of the following, determine whether the given matrix must be symmetric or could be nonsymmetric:
    (a) $A = C + C^T$ (b) $B = C - C^T$
    (c) $D = C^TC$ (d) $E = C^TC - CC^T$
    (e) $F = (I + C)(I + C^T)$
    (f) $G = (I + C)(I - C^T)$

12. Let
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
Show that if $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then
$$A^{-1} = \frac{1}{d}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

13. Use the result from Exercise 12 to find the inverse of each of the following matrices:
    (a) $\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

14. Let A and B be $n \times n$ matrices. Show that if
$$AB = A \quad \text{and} \quad B \neq I$$
then A must be singular.

15. Let A be a nonsingular matrix. Show that A^{-1} is also nonsingular and $(A^{-1})^{-1} = A$.

16. Prove that if A is nonsingular, then A^T is nonsingular and
$$(A^T)^{-1} = (A^{-1})^T$$
[*Hint*: $(AB)^T = B^TA^T$.]

17. Let A be an $n \times n$ matrix and let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$. Show that if $A\mathbf{x} = A\mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then the matrix A must be singular.