$1, x, x^2, \ldots, x^{n-1}$, and consequently, ...

Although these standard bases appear to be the simplest and most natural to use, they are not the most appropriate bases for many applied problems. Indeed, the key to solving many applied problems is to switch from one of the standard bases to a basis that is in some sense natural for the particular application. Once the application is solved in terms of the new basis, it is a simple matter to switch back and represent the solution in terms of the standard basis. In the next section, we will learn how to switch from one basis to another.

SECTION 4 EXERCISES

1. In Exercise 1 of Section 3, indicate whether the given vectors form a basis for $\mathbb{R}^2$.

2. In Exercise 2 of Section 3, indicate whether the given vectors form a basis for $\mathbb{R}^3$.

3. Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

   (a) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for $\mathbb{R}^2$.
   (b) Why must $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ be linearly dependent?
   (c) What is the dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

4. Given the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -6 \\ 4 \\ -8 \end{bmatrix}$$

   what is the dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

5. Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

   (a) Show that $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are linearly dependent.
   (b) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.
   (c) What is the dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?
   (d) Give a geometric description of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.

6. In Exercise 2 of Section 2, some of the sets formed subspaces of $\mathbb{R}^3$. In each of these cases, find a basis for the subspace and determine its dimension.

7. Find a basis for the subspace S of $\mathbb{R}^4$ consisting of all vectors of the form $(a + b, a - b + 2c, b, c)^T$, where a, b, and c are all real numbers. What is the dimension of S?

8. Given $\mathbf{x}_1 = (1, 1, 1)^T$ and $\mathbf{x}_2 = (3, -1, 4)^T$:
   (a) Do $\mathbf{x}_1$ and $\mathbf{x}_2$ span $\mathbb{R}^3$? Explain.
   (b) Let $\mathbf{x}_3$ be a third vector in $\mathbb{R}^3$ and set $X = (\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3)$. What condition(s) would X have to satisfy in order for $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ to form a basis for $\mathbb{R}^3$?
   (c) Find a third vector $\mathbf{x}_3$ that will extend the set $\{\mathbf{x}_1, \mathbf{x}_2\}$ to a basis for $\mathbb{R}^3$.

9. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be linearly independent vectors in $\mathbb{R}^3$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^2$.
   (a) Describe geometrically $\text{Span}(\mathbf{a}_1, \mathbf{a}_2)$.
   (b) If $A = (\mathbf{a}_1, \mathbf{a}_2)$ and $\mathbf{b} = A\mathbf{x}$, then what is the dimension of $\text{Span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{b})$? Explain.

10. The vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix},$$

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

   span $\mathbb{R}^3$. Pare down the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ to form a basis for $\mathbb{R}^3$.

11. Let S be the subspace of P_3 consisting of all polynomials of the form $ax^2 + bx + 2a + 3b$. Find a basis for S.

12. In Exercise 3 of Section 2, some of the sets formed subspaces of $\mathbb{R}^{2\times 2}$. In each of these cases, find a basis for the subspace and determine its dimension.

13. In $C[-\pi, \pi]$, find the dimension of the subspace spanned by $1, \cos 2x$, and $\cos^2 x$.

14. In each of the following, find the dimension of the subspace of P_3 spanned by the given vectors:
   (a) $x, x - 1, x^2 + 1$