(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(e) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$

5. For each consistent system in Exercise 4, determine whether there will be one or infinitely many solutions by examining the column vectors of the coefficient matrix A.

6. How many solutions will the linear system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is in the column space of A and the column vectors of A are linearly dependent? Explain.

7. Let A be an $6 \times n$ matrix of rank r and let $\mathbf{b}$ be a vector in $\mathbb{R}^6$. For each pair of values of r and n that follow, indicate the possibilities as to the number of solutions one could have for the linear system $A\mathbf{x} = \mathbf{b}$. Explain your answers.

   (a) $n = 7, r = 5$ (b) $n = 7, r = 6$
   (c) $n = 5, r = 5$ (d) $n = 5, r = 4$

8. Let A be an $m \times n$ matrix with $m > n$. Let $\mathbf{b} \in \mathbb{R}^m$ and suppose that $N(A) = \{\mathbf{0}\}$.
   (a) What can you conclude about the column vectors of A? Are they linearly independent? Do they span $\mathbb{R}^m$? Explain.
   (b) How many solutions will the system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is not in the column space of A? How many solutions will there be if $\mathbf{b}$ is in the column space of A? Explain.

9. Let A and B be 6×5 matrices. If $\dim N(A) = 2$, what is the rank of A? If the rank of B is 4, what is the dimension of $N(B)$?

10. Let A be an $m \times n$ matrix whose rank is equal to n. If $A\mathbf{c} = A\mathbf{d}$, does this imply that $\mathbf{c}$ must be equal to $\mathbf{d}$? What if the rank of A is less than n? Explain your answers.

11. Let A be an $m \times n$ matrix. Prove that

$$\text{rank}(A) \leq \min(m, n)$$

12. Let A and B be row-equivalent matrices.
   (a) Show that the dimension of the column space of A equals the dimension of the column space of B.
   (b) Are the column spaces of the two matrices necessarily the same? Justify your answer.

13. Let A be a 4×3 matrix and suppose that the vectors

$$\mathbf{z}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{z}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

form a basis for $N(A)$. If $\mathbf{b} = \mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3$, find all solutions of the system $A\mathbf{x} = \mathbf{b}$.

14. Let A be a 4×4 matrix with reduced row echelon form given by

$$U = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If

$$\mathbf{a}_1 = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{bmatrix} 4 \\ -3 \\ 7 \\ -1 \end{bmatrix}$$

find $\mathbf{a}_3$ and $\mathbf{a}_4$.

15. Let A be a 4×5 matrix and let U be the reduced row echelon form of A. If

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) find a basis for $N(A)$.
   (b) given that $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

      (i) find all solutions to the system.
      (ii) determine the remaining column vectors of A.

16. Let A be a 5×8 matrix with rank equal to 5 and let $\mathbf{b}$ be any vector in $\mathbb{R}^5$. Explain why the system $A\mathbf{x} = \mathbf{b}$ must have infinitely many solutions.