13. Let L be the linear transformation mapping P_2 into $\mathbb{R}^2$ defined by

$$L(p(x)) = \begin{bmatrix} \int_0^1 p(x)\,dx \\ p(0) \end{bmatrix}$$

Find a matrix A such that

$$L(\alpha + \beta x) = A \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

14. The linear transformation L defined by

$$L(p(x)) = p'(x) + p(0)$$

maps P_3 into P_2. Find the matrix representation of L with respect to the ordered bases $[x^2, x, 1]$ and $[2, 1 - x]$. For each of the following vectors $p(x)$ in P_3, find the coordinates of $L(p(x))$ with respect to the ordered basis $[2, 1 - x]$:

(a) $x^2 + 2x - 3$ (b) $x^2 + 1$

(c) $3x$ (d) $4x^2 + 2x$

15. Let S be the subspace of $C[a, b]$ spanned by e^x, xe^x, and x^2e^x. Let D be the differentiation operator of S. Find the matrix representing D with respect to $[e^x, xe^x, x^2e^x]$.

16. Let L be a linear operator on $\mathbb{R}^n$. Suppose that $L(\mathbf{x}) = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$. Let A be the matrix representing L with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Show that A is singular.

17. Let L be a linear operator on a vector space V. Let A be the matrix representing L with respect to the ordered basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V, that is, $L(\mathbf{v}_j) = \sum_{i=1}^n a_{ij}\mathbf{v}_i$, $j = 1, \ldots, n$. Show that A^m is the matrix representing L^m with respect to $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

18. Let $E = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $F = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{b}_1 = (1, -1)^T, \quad \mathbf{b}_2 = (2, -1)^T$$

For each of the following linear transformations L from $\mathbb{R}^3$ into $\mathbb{R}^2$, find the matrix representing L with respect to the ordered bases E and F:

(a) $L(\mathbf{x}) = (x_3, x_1)^T$

(b) $L(\mathbf{x}) = (x_1 + x_2, x_1 - x_3)^T$

(c) $L(\mathbf{x}) = (2x_2, -x_1)^T$

19. Suppose that $L_1 \colon V \to W$ and $L_2 \colon W \to Z$ are linear transformations and E, F, and G are ordered bases for V, W, and Z, respectively. Show that, if A represents L_1 relative to E and F and B represents L_2 relative to F and G, then the matrix $C = BA$ represents $L_2 \circ L_1 \colon V \to Z$ relative to E and G. [*Hint*: Show that $BA[\mathbf{v}]_E = [(L_2 \circ L_1)(\mathbf{v})]_G$ for all $\mathbf{v} \in V$.]

20. Let V and W be vector spaces with ordered bases E and F, respectively. If $L \colon V \to W$ is a linear transformation and A is the matrix representing L relative to E and F, show that

(a) $\mathbf{v} \in \ker(L)$ if and only if $[\mathbf{v}]_E \in N(A)$.

(b) $\mathbf{w} \in L(V)$ if and only if $[\mathbf{w}]_F$ is in the column space of A.