SECTION 2 EXERCISES

1. For each of the following matrices, determine a basis for each of the subspaces $R(A^T)$, $N(A)$, $R(A)$, and $N(A^T)$:

(a) $A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$ **(d)** $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}$

2. Let S be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{x} = (1, -1, 1)^T$.

(a) Find a basis for $S^\perp$.

(b) Give a geometrical description of S and $S^\perp$.

3. (a) Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{y} = (y_1, y_2, y_3)^T$. Let

$$A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

Show that $S^\perp = N(A)$.

(b) Find the orthogonal complement of the subspace of $\mathbb{R}^3$ spanned by $(1, 2, 1)^T$ and $(1, -1, 2)^T$.

4. Let S be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1 = (1, 0, -2, 1)^T$ and $\mathbf{x}_2 = (0, 1, 3, -2)^T$. Find a basis for $S^\perp$.

5. Let A be a 3×2 matrix with rank 2. Give geometric descriptions of $R(A)$ and $N(A^T)$, and describe geometrically how the subspaces are related.

6. Is it possible for a matrix to have the vector $(3, 1, 2)$ in its row space and $(2, 1, 1)^T$ in its null space? Explain.

7. Let $\mathbf{a}_j$ be a nonzero column vector of an $m \times n$ matrix A. Is it possible for $\mathbf{a}_j$ to be in $N(A^T)$? Explain.

8. Let S be the subspace of $\mathbb{R}^n$ spanned by the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Show that $\mathbf{y} \in S^\perp$ if and only if $\mathbf{y} \perp \mathbf{x}_i$ for $i = 1, \ldots, k$.

9. If A is an $m \times n$ matrix of rank r, what are the dimensions of $N(A)$ and $N(A^T)$? Explain.

10. Prove Corollary 2.5.

11. Prove: If A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$, then either $A\mathbf{x} = \mathbf{0}$ or there exists $\mathbf{y} \in R(A^T)$ such that $\mathbf{x}^T\mathbf{y} \neq 0$. Draw a picture similar to Figure 2.2 to illustrate this result geometrically for the case where $N(A)$ is a two-dimensional subspace of $\mathbb{R}^3$.