## SECTION 3 EXERCISES

**1.** Find the least squares solution of each of the following systems:

(a) $x_1 + x_2 = 3$
$2x_1 - 3x_2 = 1$
$0x_1 + 0x_2 = 2$

(b) $-x_1 + x_2 = 10$
$2x_1 + x_2 = 5$
$x_1 - 2x_2 = 20$

(c) $x_1 + x_2 + x_3 = 4$
$-x_1 + x_2 + x_3 = 0$
$- x_2 + x_3 = 1$
$x_1 + x_3 = 2$

**2.** For each of your solutions $\hat{\mathbf{x}}$ in Exercise 1,

(a) determine the projection $\mathbf{p} = A\hat{\mathbf{x}}$.

(b) calculate the residual $r(\hat{\mathbf{x}})$.

(c) verify that $r(\hat{\mathbf{x}}) \in N(A^T)$.

**3.** For each of the following systems $A\mathbf{x} = \mathbf{b}$, find all least squares solutions:

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 8 \end{bmatrix}$

**4.** For each of the systems in Exercise 3, determine the projection $\mathbf{p}$ of $\mathbf{b}$ onto $R(A)$ and verify that $\mathbf{b} - \mathbf{p}$ is orthogonal to each of the column vectors of $A$.

**5.** (a) Find the best least squares fit by a linear function to the data

| $x$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|
| $y$ | $0$ | $1$ | $3$ | $9$ |

(b) Plot your linear function from part (a) along with the data on a coordinate system.

**6.** Find the best least squares fit to the data in Exercise 5 by a quadratic polynomial. Plot the points $x = -1, 0, 1, 2$ for your function and sketch the graph.

**7.** Given a collection of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, let

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n)^T$$

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

and let $y = c_0 + c_1 x$ be the linear function that gives the best least squares fit to the points. Show that if $\overline{x} = 0$, then

$$c_0 = \overline{y} \quad \text{and} \quad c_1 = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{x}^T\mathbf{x}}$$

**8.** The point $(\overline{x}, \overline{y})$ is the *center of mass* for the collection of points in Exercise 7. Show that the least squares line must pass through the center of mass. [*Hint*: Use a change of variables $z = x - \overline{x}$ to translate the problem so that the new independent variable has mean 0.]

**9.** Let $A$ be an $m \times n$ matrix of rank $n$ and let $P = A(A^TA)^{-1}A^T$.

(a) Show that $P\mathbf{b} = \mathbf{b}$ for every $\mathbf{b} \in R(A)$. Explain this property in terms of projections.

(b) If $\mathbf{b} \in R(A)^\perp$, show that $P\mathbf{b} = \mathbf{0}$.

(c) Give a geometric illustration of parts (a) and (b) if $R(A)$ is a plane through the origin in $\mathbb{R}^3$.

**10.** Let $A$ be an $8 \times 5$ matrix of rank 3, and let $\mathbf{b}$ be a nonzero vector in $N(A^T)$.

(a) Show that the system $A\mathbf{x} = \mathbf{b}$ must be inconsistent.

(b) How many least squares solutions will the system $A\mathbf{x} = \mathbf{b}$ have? Explain.

**11.** Let $P = A(A^TA)^{-1}A^T$, where $A$ is an $m \times n$ matrix of rank $n$.

(a) Show that $P^2 = P$.

(b) Prove that $P^k = P$ for $k = 1, 2, \ldots$.

(c) Show that $P$ is symmetric. [*Hint*: If $B$ is nonsingular, then $(B^{-1})^T = (B^T)^{-1}$.]

**12.** Show that if

$$\begin{bmatrix} A & I \\ O & A^T \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

then $\hat{\mathbf{x}}$ is a least squares solution of the system $A\mathbf{x} = \mathbf{b}$ and $\mathbf{r}$ is the residual vector.

**13.** Let $A \in \mathbb{R}^{m\times n}$ and let $\hat{\mathbf{x}}$ be a solution of the least squares problem $A\mathbf{x} = \mathbf{b}$. Show that a vector $\mathbf{y} \in \mathbb{R}^n$ will also be a solution if and only if $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$, for some vector $\mathbf{z} \in N(A)$. [*Hint*: $N(A^TA) = N(A)$.]

**14.** Find the equation of the circle that gives the best least squares circle fit to the points $(-1, -2)$, $(0, 2.4)$, $(1.1, -4)$, and $(2.4, -1.6)$.