SECTION 5 EXERCISES

1. Which of the following sets of vectors form an orthonormal basis for $\mathbb{R}^2$?

(a) $\{(1, 0)^T, (0, 1)^T\}$

(b) $\left\{\left(\frac{3}{5}, \frac{4}{5}\right)^T, \left(\frac{5}{13}, \frac{12}{13}\right)^T\right\}$

(c) $\{(1, -1)^T, (1, 1)^T\}$

(d) $\left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T, \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T\right\}$

2. Let

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Let $\mathbf{x} = (1, 1, 1)^T$. Write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ using Theorem 5.2 and use Parseval's formula to compute $\|\mathbf{x}\|$.

3. Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ of Exercise 2. Let $\mathbf{x} = (1, 2, 2)^T$. Find the projection $\mathbf{p}$ of $\mathbf{x}$ onto S. Show that $(\mathbf{p} - \mathbf{x}) \perp \mathbf{u}_2$ and $(\mathbf{p} - \mathbf{x}) \perp \mathbf{u}_3$.

4. Let θ be a fixed real number and let

$$\mathbf{x}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

(a) Show that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

(b) Given a vector $\mathbf{y}$ in $\mathbb{R}^2$, write it as a linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

(c) Verify that

$$c_1^2 + c_2^2 = \|\mathbf{y}\|^2 = y_1^2 + y_2^2$$

5. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthonormal basis for $\mathbb{R}^2$ and let $\mathbf{u}$ be a unit vector in $\mathbb{R}^2$. If $\mathbf{u}^T\mathbf{u}_1 = \frac{1}{2}$, determine the value of $|\mathbf{u}^T\mathbf{u}_2|$.

6. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product space V and let

$$\mathbf{u} = \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3 \quad \text{and} \quad \mathbf{v} = \mathbf{u}_1 + 7\mathbf{u}_3$$

Determine the value of each of the following:

(a) $\langle \mathbf{u}, \mathbf{v} \rangle$ **(b)** $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$

(c) The angle θ between $\mathbf{u}$ and $\mathbf{v}$

7. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product space V. If $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ is a vector with the properties $\|\mathbf{x}\| = 5$, $\langle \mathbf{u}_1, \mathbf{x} \rangle = 4$, and $\mathbf{x} \perp \mathbf{u}_2$, then what are the possible values of c_1, c_2, and c_3?

8. The functions $\cos x$ and $\sin x$ form an orthonormal set in $C[-\pi, \pi]$. If

$$f(x) = 3\cos x + 2\sin x \quad \text{and} \quad g(x) = \cos x - \sin x$$

use Corollary 5.3 to determine the value of

$$\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$$

9. The set

$$S = \left\{\frac{1}{\sqrt{2}}, \cos x, \cos 2x, \cos 3x, \cos 4x\right\}$$

is an orthonormal set of vectors in $C[-\pi, \pi]$ with inner product defined by (2).

(a) Use trigonometric identities to write the function $\sin^4 x$ as a linear combination of elements of S.