SECTION 1 EXERCISES

1. Find the eigenvalues and the corresponding eigenspaces for each of the following matrices:

(a) $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$

(i) $\begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ (j) $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

(k) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ (l) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

2. Show that the eigenvalues of a triangular matrix are the diagonal elements of the matrix.

3. Let A be an $n \times n$ matrix. Prove that A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

4. Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

5. Let A and B be $n \times n$ matrices. Show that if none of the eigenvalues of A are equal to 1, then the matrix equation

$$XA + B = X$$

will have a unique solution.

6. Let λ be an eigenvalue of A and let $\mathbf{x}$ be an eigenvector belonging to λ. Use mathematical induction to show that, for $m \geq 1$, λ^m is an eigenvalue of A^m and $\mathbf{x}$ is an eigenvector of A^m belonging to λ^m.

7. Let A be an $n \times n$ matrix and let $B = I - 2A + A^2$.
   (a) Show that if $\mathbf{x}$ is an eigenvector of A belonging to an eigenvalue λ of A, then $\mathbf{x}$ is also an eigenvector of B belonging to an eigenvalue μ of B. How are λ and μ related?
   (b) Show that if $\lambda = 1$ is an eigenvalue of A, then the matrix B will be singular.

8. An $n \times n$ matrix A is said to be *idempotent* if $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix, then λ must be either 0 or 1.

9. An $n \times n$ matrix is said to be *nilpotent* if $A^k = O$ for some positive integer k. Show that all eigenvalues of a nilpotent matrix are 0.

10. Let A be an $n \times n$ matrix and let $B = A - \alpha I$ for some scalar α. How do the eigenvalues of A and B compare? Explain.

11. Let A be an $n \times n$ matrix and let $B = A + I$. Is it possible for A and B to be similar? Explain.

12. Show that A and A^T have the same eigenvalues. Do they necessarily have the same eigenvectors? Explain.

13. Show that the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

will have complex eigenvalues if θ is not a multiple of π. Give a geometric interpretation of this result.

14. Let A be a 2×2 matrix. If $\text{tr}(A) = 8$ and $\det(A) = 12$, what are the eigenvalues of A?

15. Let $A = (a_{ij})$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that

$$\lambda_j = a_{jj} + \sum_{i \neq j} (a_{ii} - \lambda_i) \quad \text{for } j = 1, \ldots, n$$