3 Exercises

In Exercises 1–4, verify Stokes's theorem for the given surface and vector field.

1. S is defined by $x^2 + y^2 + 5z = 1$, $z \geq 0$, oriented by upward normal;

$$\mathbf{F} = xz\,\mathbf{i} + yz\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k}.$$

2. S is parametrized by $\mathbf{X}(s, t) = (s \cos t, s \sin t, t)$, $0 \leq s \leq 1$, $0 \leq t \leq \pi/2$;

$$\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}.$$

3. S is defined by $x = \sqrt{16 - y^2 - z^2}$;

$$\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}.$$

4. S is defined by $x^2 + y^2 + z^2 = 4$, $z \leq 0$, oriented by downward normal;

$$\mathbf{F} = (2y - z)\,\mathbf{i} + (x + y^2 - z)\,\mathbf{j} + (4y - 3x)\,\mathbf{k}.$$

5. Let S be the "silo surface," that is, S is the union of two smooth surfaces S_1 and S_2, where S_1 is defined by

$$x^2 + y^2 = 9, \quad 0 \leq z \leq 8$$

and S_2 is defined by

$$x^2 + y^2 + (z - 8)^2 = 9, \quad z \geq 8.$$

Find $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = (x^3 + xz + yz^2)\,\mathbf{i} + (xyz^3 + y^7)\,\mathbf{j} + x^2 z^5\,\mathbf{k}.$$

In Exercises 6–9, verify Gauss's theorem for the given three-dimensional region D and vector field **F**.

6. $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $D = \{(x, y, z) \mid 0 \leq z \leq 9 - x^2 - y^2\}$

7. $\mathbf{F} = (y - x)\,\mathbf{i} + (y - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$, D is the unit cube $[0, 1] \times [0, 1] \times [0, 1]$

8. $\mathbf{F} = x^2\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $D = \{(x, y, z) \mid x^2 + y^2 + 1 \leq z \leq 5\}$

9. $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$, $D = \{(x, y, z) \mid a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$

10. Verify that Stokes's theorem implies Green's theorem. (Hint: In Stokes's theorem take $\mathbf{F}(x, y, z) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$; that is, assume $\mathbf{F}$ is independent of z and that its $\mathbf{k}$-component is identically zero.)

11. Let S be the surface defined by $y = 10 - x^2 - z^2$ with $y \geq 1$, oriented with rightward-pointing normal. Let

$$\mathbf{F} = (2xyz + 5z)\,\mathbf{i} + e^x \cos yz\,\mathbf{j} + x^2 y\,\mathbf{k}.$$

Determine

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

(Hint: You will need an indirect approach.)

12. Let S be the surface defined as $z = 4 - 4x^2 - y^2$ with $z \geq 0$ and oriented by a normal with nonnegative $\mathbf{k}$-component. Let $\mathbf{F}(x, y, z) = x^3\mathbf{i} + e^{y^2}\mathbf{j} + ze^{xy}\mathbf{k}$. Find $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$. (Hint: Argue that you can integrate over a different surface.)

13. (a) Show that the path $\mathbf{x}(t) = (\cos t, \sin t, \sin 2t)$ lies on the surface $z = 2xy$.

(b) Evaluate

$$\oint_C (y^3 + \cos x)\,dx + (\sin y + z^2)\,dy + x\,dz,$$

where C is the closed curve parametrized and oriented by the path $\mathbf{x}$ in part (a).