Applied Finite Mathematics

Tomastik/Epstein

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Applied Finite Mathematics, Second Edition

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Circuit Boards

How should the circuits on this board be laid out so that the video card works? Logic is used in the design of circuit boards.

L.1   Introduction to Logic

✧ Statements

Logic is the science of correct reasoning and of making valid conclusions. In logic conclusions must be inescapable. Every concept must be clearly defined. Thus, dictionary definitions are often not sufficient since there can be no ambiguities or vagueness.

We restrict our study to declarative sentences that are unambiguous and that can be classified as true or false but not both. Such declarative sentences are called **statements** and form the basis of logic.

> **Statements**
> A **statement** is a declarative sentence that is either true or false but not both.

Thus, commands, questions, exclamations, and ambiguous sentences cannot be statements.

EXAMPLE 1   **Determining if Sentences Are Statements**   Decide which of the following sentences are statements and which are not.

a. Look at me.
b. Do you enjoy music?
c. What a beautiful sunset!
d. Two plus two equals four.
e. Two plus two equals five.
f. The author got out of bed after 6:00 A.M. today.
g. That was a great game.
h. $x + 2 = 5$.

Solution   The first three sentences are not statements since the first is a command, the second is a question, and the third is an exclamation. Sentences **d** and **e** are statements; **d** is a true statement while **e** is a false statement. Sentence **f** is a statement, but you do not know if it is true or not. Sentence **g** is not a statement since we are not told what "great" means. With a definition of great, such as "Our team won," then it would be a statement. The last sentence **h.** is not a statement since it cannot be classified as true or false. For example, if $x = 3$ it is true. But if $x = 2$ it is false. ✦

✧ Connectives

A statement such as "I have money in my pocket" is called a **simple** statement since it expresses a single thought. But we need to also deal with **compound** statements such as "I have money in my pocket and my gas tank is full." We will let letters such as p, q, and r denote simple statements. To write compound statements, we need to introduce symbols for the **connectives**.

HISTORICAL NOTE

A Brief History of Logic

The Greek philosopher Aristotle (384–322 B.C.) is generally given the credit for the first systemic study of logic. His work, however, used ordinary language. The second great period of logic came with Gottfried Leibnitz (1646–1716), who initiated the use of symbols to simplify complicated logical arguments. This treatment is referred to as **symbolic logic** or **mathematical logic**. In symbolic logic, symbols and prescribed rules are used very much as in ordinary algebra. This frees the subject from the ambiguities of ordinary language and permits the subject to proceed and develop in a methodical way. It was, however, Augustus De Morgan (1806–1871) and George Boole (1815–1864) who systemically developed symbolic logic. The "algebra" of logic that they developed removed logic from philosophy and attached it to mathematics.

> **Connectives**
>
> A **connective** is a word or words, such as "and" or "if and only if," that is used to combine two or more simple statements into a compound statement.

We will consider the 5 connectives given in the following table. We will discuss the first 3 in this section and the last 2 in the third section of this chapter.

Name	Connective	Symbol
Conjunction	and	$\wedge$
Disjunction	or	$\vee$
Negation	not	$\sim$
Conditional	if … then	$\rightarrow$
Biconditional	if and only if	$\leftrightarrow$

Logic does not concern itself with whether a simple statement is true or false. But if all the simple statements that make up a compound statement are known to be true or false, then the rules of logic will enable us to determine if the compound statement is true or false. We will do this in the next section.

We now carefully give the definitions of the three connectives "and," "or," and "not." Notice that the precise meanings of the three compound statements that involve these connectives are incomplete unless a clear statement is made as to when the compound statement is true and when it is false. The first connective we discuss is **conjunction** which is the concept of "and."

> **Conjunction**
>
> A **conjunction** is a statement of the form "p and q" and is written symbolically as
> $$p \wedge q$$
> The conjunction $p \wedge q$ is true if both p and q are true, otherwise it is false.

EXAMPLE 2   **Using Conjunction**   Write the compound statement "I have money in my pocket and my gas tank is full" in symbolic form.

Solution   First let p be the statement "I have money in my pocket" and q be the statement "my gas tank is full." Since $\wedge$ represents the word "and," the compound statement can be written symbolically as $p \wedge q$.   ✦

The next connective we consider is **disjunction** which is the concept of "or." Make careful note of the fact that the logical "or" is slightly different in meaning than the typical English use of the word "or."

> **Disjunction**
>
> A **disjunction** is a statement of the form "p or q" and is written symbolically as
>
> $$p \vee q$$
>
> The disjunction $p \vee q$ is false if both p and q are false and is true in all other cases.

REMARK: The word "or" in this definition conveys the meaning "one or the other, or both." This is also called the **inclusive or**.

EXAMPLE 3   Using Disjunction   Write the compound statement "Janet is in the top 10% of her class or she lives on campus" in symbolic form.

Solution   First let p be the statement "Janet is in the top 10% of her class" and q the statement "She lives on campus." Since $\vee$ represents the word "or," the compound statement can be written as $p \vee q$.   ✦

REMARK: In everyday language the word "or" is not always used in the way indicated above. For example, if a car salesman tells you that for $20,000 you can have a new car with automatic transmission or a new car with air conditioning, he means "one or the other, but not both." This use of the word "or" is called **exclusive or**.

The final connective introduced in this section is **negation** which is the concept of "not."

Negation

> **Negation**
>
> A **negation** is a statement of the form "not p" and is written symbolically as
>
> $$\sim p$$
>
> The negation $\sim p$ is true if p is false and false if p is true.

For example, if p is the statement "Janet is smart," then $\sim p$ is the statement "Janet is not smart."

EXAMPLE 4   Using Negation   Let p and q be the following statements:

p: George Bush plays football for the Washington Redskins.
q: The Dow Jones industrial average set a new record high last week.

Write the following statements in symbolic form.

a. George Bush does not play football for the Washington Redskins, and the Dow Jones industrial average set a new record high last week.

b. George Bush plays football for the Washington Redskins, or the Dow Jones industrial average did not set a new record high last week.

c. George Bush does not play football for the Washington Redskins, and the Dow Jones industrial average did not set a new record high last week.

d. It is not true that George Bush plays football for the Washington Redskins and that the Dow Jones industrial average set a new record high last week.

Solution **a.** $(\sim p) \wedge q$ **b.** $p \vee \sim q$ **c.** $\sim p \wedge \sim q$ **d.** $\sim (p \wedge q)$ ✦

EXAMPLE 5 **Translating Symbolic Forms Into Compound Statements** Let p and q be the following statements:

p: Philadelphia is the capital of New Jersey.
q: General Electric lost money last year.

Write out the statements that correspond to each of the following:
a. $p \vee q$ **b.** $p \wedge q$ **c.** $p \vee \sim q$ **d.** $\sim p \wedge \sim q$

Solution

a. Philadelphia is the capital of New Jersey, or General Electric lost money last year.

b. Philadelphia is the capital of New Jersey, and General Electric lost money last year.

c. Philadelphia is the capital of New Jersey, or General Electric did not lose money last year.

d. Philadelphia is not the capital of New Jersey, and General Electric did not lose money last year.

✦

In most cases when dealing with complex compound statements, there will not be a question as to the order in which to apply the connectives. However, you may have noticed in the above examples that negation was used before disjunction or conjunction. The order of precedence for logical connectives is stated below.

Order of Precedence
The logical connectives are used in the following order

$$\sim, \wedge, \vee, \rightarrow, \leftrightarrow$$

Self-Help Exercises L.1

1. Determine which of the following sentences are statements:

    a. The Atlanta Braves won the World Series in 1992.

    b. IBM makes oil tankers for Denmark.

    c. Does IBM make oil tankers for Denmark?

    d. Please pay attention.

    e. I have a three-dollar bill in my purse, or I don't have a purse.

2. Let p be the statement "George Washington was never president of the United States" and q be the

statement "George Washington wore a wig." Write out the statements that correspond to the following:

a. $\sim p$ **b.** $p \vee q$ **c.** $\sim p \wedge q$
d. $p \wedge \sim q$ **e.** $\sim p \vee \sim q$

L.1 Exercises

In Exercises 1 through 14, decide which are statements.

1. Water freezes at 70^oF.

2. It rained in St. Louis on May 4, 1992.

3. $5 > 10$.

4. This sentence is false.

5. The number 4 is not a prime.

6. How are you feeling?

7. I feel great!

8. $10 + 10 - 5 = 25$

9. There is life on Mars.

10. Cleveland is the largest city in Ohio.

11. Who said Cleveland is the largest city in Ohio?

12. You don't say!

13. IBM lost money in 1947.

14. Groundhog Day is on February 12.

15. Let p and q denote the following statements:

    p: George Washington was the third president of the United States.
    q: Austin is the capital of Texas.

    Express the following compound statements in words:

    a. $\sim p$ **b.** $p \wedge q$ **c.** $p \vee q$
    d. $\sim p \wedge q$ **e.** $p \vee \sim q$ **f.** $\sim (p \wedge q)$

16. Let p and q denote the following statements:

    p: Mount McKinley is the highest point in the United States.
    q: George Washington was a signer of the Declaration of Independence.

    Express the following compound statements in words.

    a. $\sim q$ **b.** $p \wedge q$ **c.** $p \vee q$
    d. $p \wedge \sim q$ **e.** $\sim p \wedge \sim q$ **f.** $\sim (p \vee q)$

17. Let p and q denote the following statements:

    p: George Washington owned over 100,000 acres of property.
    q: The Exxon Valdez was a luxury liner.

    a. State the negation of these statements in words.
    b. State the disjunction for these statements in words.
    c. State the conjunction for these statements in words.

18. Let p and q denote the following statements:

    p: McDonald's Corporation operates large farms.
    q: Wendy's Corporation operates fast-food restaurants.

    a. State the negation of these statements in words.
    b. State the disjunction for these statements in words.
    c. State the conjunction for these statements in words.

19. Let p and q denote the following statements:

    p: The *Wall Street Journal* has the highest daily circulation of any newspaper.
    q: *Advise and Consent* was written by Irving Stone.

    Give a symbolic expression for the statements below.

    a. *Advise and Consent* was not written by Irving Stone.
    b. The *Wall Street Journal* has the highest daily circulation of any newspaper, and *Advise and Consent* was not written by Irving Stone.
    c. The *Wall Street Journal* has the highest daily circulation of any newspaper, or *Advise and Consent* was written by Irving Stone.

d. The *Wall Street Journal* does not have the highest daily circulation of any newspaper, or *Advise and Consent* was not written by Irving Stone.

20. Let *p* and *q* denote the following statements:

p: IBM makes computers.

q: IBM makes trucks.

Give a symbolic expression for the statements below.

a. IBM does not make trucks.

b. IBM makes computers, or IBM makes trucks.

c. IBM makes computers, or IBM does not make trucks.

d. IBM does not make computers, and IBM does not make trucks.

Solutions to Self-Help Exercises L.1

1. The sentences **a**, **b**, and **e** are statements, while **c** and **d** are not.

2. **a.** George Washington was a president of the United States.

b. George Washington was never president of the United States, or George Washington wore a wig.

c. George Washington was a president of the United States, and George Washington wore a wig.

d. George Washington was never president of the United States, and George Washington did not wear a wig.

e. George Washington was a president of the United States, or George Washington did not wear a wig.

L.2   Truth Tables

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table L.1

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table L.2

p	$\sim p$
T	F
F	T

Table L.3

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Table L.4

✧ Introduction to Truth Tables

The **truth value** of a statement is either true or false. Thus the statement "Ronald H. Coarse won the Nobel Prize in Economics in 1991" has truth value true since it is a true statement, whereas the statement "Los Angeles is the capital of California" has truth value false since it is a false statement.

Logic does not concern itself with the truth value of simple statements. But if we know the truth values of the simple statements that make up a compound statement, then logic can determine the truth value of the compound statement.

For example, to understand the very definition of $p \vee q$, one must know under what conditions the compound statement will be true. As defined in the last section $p \vee q$ is always true unless both p and q are false. A convenient way of summarizing this is by a truth table. This is done in Table L.1.

The truth tables for the statements $p \wedge q$ and $\sim p$ are given in Table L.2 and Table L.3. As Table L.2 indicates, $p \wedge q$ is true only if both p and q are true. Given a general compound statement, we wish to determine the truth value given any possible combination of truth values for the simple statements that are contained in the compound statement. We use a truth table for this purpose. The next examples illustrate how this is done.

EXAMPLE 1   **Constructing a Truth Table**   Construct a truth table for the statement $p \vee \sim q$.

Solution   Place p and q at the head of the first two columns as indicated in Table L.4 $\vee$ found in Table L.1.   ✦ and list all possible truth values for p and q as indicated. It is strongly recommended that you always list the truth values in the first two columns in the same way. This will be particularly useful later when we will need to compare two truth tables. Now enter the truth values for $\sim q$ in the third column. Now using the first and third columns of the table, construct the fourth column using the definition of

EXAMPLE 2   **Constructing a Truth Table**   Construct a truth table for the statement $\sim p \wedge (p \vee q)$.

Solution   Make the same first two columns as before. Next make a column for $\sim p$ and the corresponding truth values. Now make a fourth column for $p \vee q$. Finally, using the third and fourth columns and the definition of $\wedge$, fill in the fifth column of Table L.5.

p	q	$\sim p$	$p \vee q$	$\sim p \wedge (p \vee q)$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

Table L.5

Thus we see that $\sim p \wedge (p \vee q)$ is true only if p is false and q is true.   ✦

We can construct a truth table for a compound statement with three simple statements.

EXAMPLE 3 **Constructing a Truth Table** Construct a truth table for the statement $(p \wedge q) \wedge [(r \vee \sim p) \wedge q]$.

Solution Always use the same order of T's and F's that are indicated in the first three columns of Table L.6. Fill in the rest of the columns in the order given.

p	q	r	$p \wedge q$	$\sim p$	$r \vee \sim p$	$(r \vee \sim p) \wedge q$	$(p \wedge q) \wedge [(r \vee \sim p) \wedge q]$
T	T	T	T	F	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	T	F	T	T	T	F
F	T	F	F	T	T	T	F
F	F	T	F	T	T	F	F
F	F	F	F	T	T	F	F

Table L.6

We see that $(p \wedge q) \wedge [(r \vee \sim p) \wedge q]$ is true only if p, q, and r are all true. ✦

✦ Exclusive Disjunction

We now consider the exclusive "or." Recall that the exclusive "or" means "one or the other, but not both." The truth table for the exclusive disjunction is given in Table L.7 where we note that the symbol for the exclusive disjunction is $\underline{\vee}$. Notice that $\underline{\vee}$ is true only if exactly one of the two statements is true.

p	q	$p \underline{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

Table L.7

REMARK: Unless clearly specified otherwise, the word "or" will always be taken in the *in*clusive sense.

EXAMPLE 4 **Determining the Truth Value of a Statement** Let p and q be the following statements:

    p: Aaron Copland was an American composer.
    q: Rudolf Serkin was a violinist.

Determine the truth value of each of the following statements:
a. $p \vee q$ **b.** $\sim (p \vee q)$ **c.** $p \underline{\vee} q$ **d.** $\sim (p \underline{\vee} q)$ **e.** $p \wedge \sim q$

Solution First note that p is true and q is false.[1] Both the disjunction in **a** and the exclusive disjunction in **c** are therefore true. Thus their negations in **b** and **d** are false. The statement in **e** is the conjunction of a true statement p with a true statement $\sim q$ and thus is true. ✦

✦ Tautology and Contradiction

The statement $p \wedge \sim p$ is always false according to the truth table in Table L.8. In such a case, we say that the statement $p \wedge \sim p$ is a **contradiction**. If the statement is always true, we say that the statement is a **tautology**.

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Table L.8

---

[1] Serkin was a famous pianist.

> **Contradiction and Tautology**
> We say that a statement is a **contradiction** if the truth value of the statement is always false no matter what the truth values of its simple component statements. We say that a statement is a **tautology** if the truth value of the statement is always true no matter what the truth values of its simple component statements.

EXAMPLE 5   Determining if a Statement Is a Tautology   Determine if the statement $p \wedge (\sim p \wedge q)$ is a tautology.

Solution   Create a truth table.

p	q	$\sim p$	$\sim p \vee q$	$p \wedge (\sim p \vee q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

The truth table indicates that the statement is true no matter what the truth values of p and q are. Thus, this statement is a tautology.   ✦

Self-Help Exercises L.2

1. Construct the truth table for the statement

$$(p \vee q) \vee (r \wedge \sim q)$$

2. Let p be the statement "George Washington was the first president of the United States" and q be the statement "George Washington wore a wig." Determine the truth value of each of the statements below.

    a. $\sim p$ **b.** $p \vee q$ **c.** $\sim p \wedge q$

    d. $p \wedge \sim q$ **e.** $\sim p \vee \sim q$

L.2 Exercises

In Exercises 1 through 20, construct a truth table for the given statement. Indicate if a statement is a tautology or a contradiction.

1. $p \wedge \sim q$

2. $p \veebar \sim q$

3. $\sim (\sim p)$

4. $\sim (p \wedge q)$

5. $(p \wedge \sim q) \vee q$

6. $(p \veebar q) \vee \sim q$

7. $\sim p \vee (p \wedge q)$

8. $\sim p \veebar (p \wedge q)$

9. $(p \vee q) \wedge (p \wedge q)$

10. $(p \wedge q) \vee (p \vee q)$

11. $(p \vee \sim q) \vee (\sim p \wedge q)$

12. $(p \wedge \sim q) \wedge (p \vee \sim q)$

13. $(p \vee q) \wedge r$

14. $p \vee (q \wedge r)$

15. $\sim [(p \wedge q) \wedge r]$

16. $\sim [p \wedge (q \wedge r)]$

17. $(p \vee q) \vee (q \wedge r)$

18. $(p \wedge q) \wedge (q \vee r)$

19. $(p \vee \sim q) \vee (\sim q \wedge r)$

20. $(\sim p \wedge q) \wedge (\sim q \veebar r)$

21. Let p and q be the statements:

    p: Roe v. Wade was a famous boxing match.

    q: Iraq invaded Kuwait in 1990.

Determine the truth value of the following compound statements:

a. $\sim p$ **b.** $p \wedge q$ **c.** $p \vee q$

d. $\sim p \wedge q$ **e.** $p \vee \sim q$

Note that p is false and q is true.

22. Let p and q be the statements:

p: The sun rises in the east.

q: Proctor & Gamble is a casino in Las Vegas.

Determine the truth value of the following compound statements:

a. $\sim q$ **b.** $p \wedge q$ **c.** $p \underline{\vee} q$

d. $p \vee \sim q$ **e.** $\sim p \vee q$

23. Let p and q be the statements:

p: The South Pole is the southernmost point on the Earth.

q: The North Pole is a monument in Washington, D.C.

Determine the truth value of the following compound statements:

a. $\sim q$ **b.** $p \underline{\vee} \sim q$ **c.** $\sim p \wedge q$

d. $\sim (p \wedge q)$

24. Let p and q be the statements:

p: Stevie Wonder is a famous singer.

q: Simon & Garfunkel is a famous law firm.

Determine the truth value of the following compound statements:

a. $p \wedge \sim q$ **b.** $p \underline{\vee} q$ **c.** $p \vee q$

d. $\sim (p \vee q)$

Note that p is true and q is false.

Solutions to Self-Help Exercises L.2

1.

p	q	r	$p \vee q$	$\sim q$	$r \wedge \sim q$	$(p \vee q) \vee (r \wedge \sim q)$
T	T	T	T	F	F	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	F	F	T
F	T	F	T	F	F	T
F	F	T	F	T	T	T
F	F	F	F	T	F	F

2. The statement p is true, while q is false. Thus

a. $\sim p$ is false.   **b.** $p \vee q$ is true.   **c.** $\sim p \wedge q$ is false.

d. $p \wedge \sim q$ is true is $\sim q$ is true.

e. $\sim p \vee \sim q$ is true.

L.3   Implication and Equivalence

✧ Conditional Statements

Statements such as "If you do all your homework, then you will pass the course" and "If you work hard, then you will succeed" are familiar. They involve the **conditional** "if ... , then." We will see that this connective is used as a basis of deductive reasoning.

> **Conditional Statement**
>
> A conditional statement is a compound statement of the form "if p, then q" and is written symbolically as
>
> $$p \rightarrow q$$
>
> A conditional statement is false if p is true and q is false and is true in all other cases. See Table L.9.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table L.9

> **Hypothesis and Conclusion**
>
> In the conditional statement $p \rightarrow q$, the statement p is the hypothesis and the statement q is the conclusion.

The question naturally arises as to why the statement $p \rightarrow q$ is defined to be true whenever p is false. To see this, consider again the statement "If you do all your homework, then you will pass this course." This can be written symbolically as $p \rightarrow q$ if p is the statement "You do all your homework" and q is the statement "You pass this course." Now if, for example, you did almost all your homework, then surely you expect to pass. In such a case, p is false and q is true. You would certainly not want the statement $p \rightarrow q$ to have truth value false in such a situation. The alternative is then to give it a truth value true.

There is another reason for the definition of $p \rightarrow q$. Everyone certainly agrees with the first two lines of the truth table for $p \rightarrow q$ in the case where p is true. The case where p is false is considered in Table L.10. There are 3 possible alternative definitions of $p \rightarrow q$ by changing the last two lines of the truth table of $p \rightarrow q$ given in Table L.9.

p	q	Possible Definitions of $p \rightarrow q$		
		1	2	3
F	T	F	T	F
F	F	T	F	F

Table L.10

Notice that the third definition (the fifth column) simply gives the definition of $p \wedge q$, while the second definition (the fourth column) is the same as q. We will see later in this section that the first definition (the third column) is the definition of the connective "if and only if." Thus if $p \rightarrow q$ were defined in any of these other ways, $\rightarrow$ would not represent a new connective.

✧ *p* implies *q*

✧ *p* only if *q*

✧ *q* if *p*

✧ *q* whenever *p*

✧ *q* provided *p*

✧ *p* is sufficient for *q*

✧ *q* is necessary for *p*

✧ suppose *p*, then *q*

There are a number of ways of stating the conditional $p \to q$ in English. Some of these are listed to the left.

EXAMPLE 1   **Recognizing $p \to q$ in English**   Write each of the following as $p \to q$, identifying *p* and *q*.

a. Working hard is sufficient for passing this course.

b. To be wealthy, it is sufficient for your parents to be wealthy.

c. $x^2 = 4$, whenever $x = \pm 2$.

d. I will succeed provided I work hard.

Solution

a. This is $p \to q$, if *p* is "I work hard" and *q* is "I pass this course."

b. This is $p \to q$, if *p* is "My parents are wealthy" and *q* is "I will be wealthy."

c. This is $p \to q$, if *p* is "$x = \pm 2$" and *q* is "$x^2 = 4$."

d. This is $p \to q$, if *p* is "I work hard" and *q* is "I will succeed."   ✦

EXAMPLE 2   **Determining the Truth Value of a Conditional Statement**   Determine the truth value of each of the following statements:

a. If Alaska is a state, then Puerto Rico is also.

b. If Puerto Rico is a state, then Alaska is also.

c. $4 + 4 = 10$, whenever $2 + 2 = 6$.

d. Miami is in Georgia provided Atlanta is in Texas.

Solution   Part **a** is false since the hypothesis is true while the conclusion is false. All the other statements **b**, **c**, and **d** are all true since in each case the hypothesis is false.   ✦

EXAMPLE 3   **Constructing a Truth Table**   Construct a truth table for $(p \to q) \wedge (q \to p)$.

Solution

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Table L.11   ✦

✧ Biconditional

We now come to the last of the 5 basic connectives, the biconditional $p \leftrightarrow q$. By this we mean $p \to q$ and simultaneously $q \to p$, that is, $(p \to q) \wedge (q \to p)$. We just considered this in Example 3, where we saw that $(p \to q) \wedge (q \to p)$ is true only when *p* and *q* are both true or both false. We then make the following definition:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table L.12

> **Biconditional Statement**
>
> A biconditional statement is a compound statement of the form "p if and only if q" and is written symbolically as
>
> $$p \leftrightarrow q$$
>
> The biconditional $p \leftrightarrow q$ is true only when p and q are both true or both false. See Table L.12.

In mathematics a common alternative way of saying "if and only if" is to say "necessary and sufficient."

EXAMPLE 4   Using Necessary and Sufficient   Rewrite the statement "$x^2 = 4$ if and only if $x = \pm 2$" using *necessary and sufficient*.

Solution   "$x^2 = 4$ is necessary and sufficient for $x = \pm 2$."   ✦

✧ Logical Equivalence

If we compare the truth table for $p \leftrightarrow q$ found in Table L.12 with the truth table for $(p \rightarrow q) \wedge (q \rightarrow p)$ found in Table L.11, we see that they are identical. This should not be surprising since $p \leftrightarrow q$ means both $p \rightarrow q$ and $q \rightarrow p$. We thus see that the statements $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent.

> **Logical Equivalence**
>
> Two statements p and q are logically equivalent, denoted by
>
> $$p \Leftrightarrow q$$
>
> if they have identical truth tables.

We now will consider three logical variants of the conditional statements $p \rightarrow q$. The first is the **contrapositive**.

> **Contrapositive**
>
> The **contrapositive** of the statement "if p, then q" ($p \rightarrow q$) is of the form "if not q, then not p" and is written
>
> $$\sim q \rightarrow \sim p$$

A truth table for the contrapositive is given in Table L.13.

p	q	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Table L.13

By comparing Table L.9 with Table L.13, we can see that the conditional $p \rightarrow q$ is logically equivalent to the contrapositive $\sim q \rightarrow \sim p$.

EXAMPLE 5   Using the Contrapositive   Write the statement "If I do all my homework, then I will pass this course" using the logically equivalent contrapositive.

Solution   Let p be the statement "I did all my homework" and q be the statement "I passed this course." Then the given compound statement can be written symbolically as the conditional $p \rightarrow q$. The contrapositive is $\sim q \rightarrow \sim p$ and is written as the statement "If I failed this course, then I did not do all my homework."   ✦

We now consider the logical variant of the conditional $p \rightarrow q$ called the **converse**.

p	q	$p \rightarrow q$
T	T	T
T	F	T
F	T	F
F	F	T

Table L.14

> **Converse**
> The **converse** of the conditional statement "if p, then q" ($p \rightarrow q$) is "if q, then p" and is written
> $$q \rightarrow p$$

The truth table is given in Table L.14. Comparing Table L.14 with Table L.9 we see that the converse $q \rightarrow p$ is not logically equivalent to the conditional $p \rightarrow q$.

EXAMPLE 6   Using the Converse   Write the converse to the conditional statement "If I do all my homework, then I pass this course."

Solution   The converse of this statement is "If I passed this course, then I did all my homework."   ✦

The last example indicates why the conditional and its converse are not logically equivalent. It is reasonable to expect that doing all your homework will result in passing this course, but this course can be passed without doing *quite* all the homework.

The last logical variant of the conditional $p \rightarrow q$ that we consider is the **inverse**.

> **Inverse**
> The **inverse** of the conditional statement "if p, then q" ($p \rightarrow q$) is "if not p, then not q" and is written
> $$\sim p \rightarrow \sim q$$

The truth table for the inverse $\sim p \rightarrow \sim q$ is given in Table L.15.

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Table L.15

Notice that Table L.15 is identical to Table L.14. Thus the converse is logically equivalent to the inverse.

It is sometimes difficult to give a direct proof of a conditional statement but easy to give a proof of the contrapositive. Since the contrapositive is logically equivalent to the conditional, establishing the contrapositive will establish the conditional. This is indicated in the next example.

EXAMPLE 7   Using the Contrapositive in a Proof   Prove that if n^2 is an odd integer, then n is also an odd integer.

Solution   Let p be the statement "n^2 is odd" and q be the statement "n is odd." Then we wish to show that $p \to q$. To do this we will show that $\sim q \to \sim p$. Assume then that n is not odd, that is, assume that n is even. Then we can write n as $n = 2k$ for some integer k and

$$n^2 = (2k)^2 = 2(2k^2)$$

Since this is twice an integer, n^2 is even and thus not odd. We have shown that if n is not odd, then n^2 is not odd. Since the contrapositive $\sim q \to \sim p$ is logically equivalent to the conditional $p \to q$, we have established the result that if n^2 is an odd integer, then n is also an odd integer.   ✦

Self-Help Exercises L.3

1. Let the statements p and q be:

   p: We raise prices.

   q: Sales drop.

   Then the conditional statement $p \to q$ is "If we raise prices, then sales drop." Write the contrapositive, converse, and inverse statements.

2. Determine if the two statements $p \to q$ and $\sim p \lor q$ are logically equivalent.

L.3 Exercises

For Exercises 1 through 6, let p and q be the statements:

   p: Interest rates drop.
   q: The stock market goes up.

Write each of the following statements in English. Then write both in English and symbolically the contrapositive, inverse, and converse of the given statement.

1. $p \to q$

2. $p \to \sim q$

3. $\sim q \to \sim p$

4. $q \to p$

5. $\sim p \to \sim q$

6. $\sim p \to q$

For Exercises 7 through 14, let p and q be the statements:

p: The rate of inflation is increasing.
q: Interest rates are increasing.

Write each of the following statements in symbolic form using → or ↔:

7. If the rate of inflation is increasing, then interest rates are increasing.

8. An increasing rate of inflation is necessary for interest rates to be increasing.

9. An increasing rate of inflation is sufficient for interest rates to be increasing.

10. If interest rates are increasing, then the rate of inflation is increasing.

11. Interest rates are increasing if and only if the rate of inflation is increasing.

12. Interest rates are increasing if the rate of inflation is increasing.

13. If interest rates are not increasing, then the rate of inflation is not increasing.

14. Interest rates are increasing only if the rate of inflation is increasing.

In Exercises 15 through 22, let *p* and *q* be the statements:

p: Elvis lives.
q: Barbra Streisand is a singer.

Determine the truth value of each of the following:

15. $p \rightarrow q$ **16.** $q \rightarrow p$

17. $\sim p \rightarrow \sim q$ **18.** $\sim q \rightarrow \sim p$

19. $(p \wedge q) \rightarrow p$ **20.** $(p \vee q) \rightarrow q$

21. $p \leftrightarrow q$ **22.** $(p \vee q) \rightarrow p$

Construct a truth table for the statements in Exercises 23 through 36.

23. $(p \rightarrow q) \wedge p$ **24.** $(p \rightarrow q) \vee p$

25. $(p \rightarrow q) \vee (q \rightarrow p)$ **26.** $\sim (p \rightarrow q)$

27. $[\sim (p \vee q)] \leftrightarrow \sim p \wedge \sim q$ **28.** $\sim (p \rightarrow \sim q)$

29. $(p \rightarrow q) \rightarrow r$ **30.** $p \rightarrow (q \rightarrow r)$

31. $[p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)]$

32. $(p \vee q) \rightarrow r$

33. $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$ **34.** $(p \rightarrow q) \wedge (\sim p \vee q)$

35. $(p \vee q) \vee r$ **36.** $p \vee (q \vee r)$

In Exercises 37 through 44, determine which are tautologies.

37. $(p \wedge q) \rightarrow p$ **38.** $p \rightarrow (p \vee q)$

39. $[p \wedge (p \rightarrow q)] \rightarrow q$ **40.** $[\sim p \wedge (p \vee q)] \rightarrow q$

41. $p \rightarrow [q \rightarrow (p \wedge q)]$

42. $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

43. $(p \rightarrow q) \rightarrow [(p \vee r) \rightarrow (q \vee r)]$

44. $(p \rightarrow q) \rightarrow [(p \wedge r) \rightarrow (q \wedge r)]$

45. Prove that if n^2 is even, then *n* is even.

46. Prove that if *mn* is odd, then both *m* and *n* are odd.

Solutions to Self-Help Exercises L.3

1. Contrapositive: "If sales did not drop, then we did not raise prices."

Converse: "If sales drop, then we raise prices."

Inverse: "If we did not raise prices, then sales did not drop."

2. From the truth table we see that the two statements $p \rightarrow q$ and $\sim p \vee q$ have identical truth tables and therefore are logically equivalent.

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

L.4 Laws of Logic

✦ Logical Equivalence

We are all aware that $a(x+y) = ax + ay$ for all numbers a, x, and y. This represents a law of numbers. No matter what the numbers a, x, and y are, the two numbers $a(x+y)$ and $ax+ay$ are equal. In an analogous fashion we noticed in the last section that given any truth value for p and q, the truth values of the statements $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are the same. Thus we said that the two statements $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ were logically equivalent and wrote $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$. The symbol $\Leftrightarrow$ in logic is thus analogous to the symbol $=$ in algebra.

We already defined two statements to be logically equivalent if their truth tables are identical. We now give an alternate definition.

> **Logical Equivalence**
>
> Two statements p and q are logically equivalent, written
>
> $$p \Leftrightarrow q$$
>
> if the biconditional $p \leftrightarrow q$ is a tautology.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table L.16

To see why these two definitions are the same, recall the truth table for the biconditional shown in Table L.16. The cases when the biconditional is true are listed in Table L.17. Notice from Table L.17 that the truth tables of p and q are the same.

p	q	$p \leftrightarrow q$
T	T	T
F	F	T

Table L.17

Laws of Logic

In the following, t is a tautology and c is a contradiction.

1.	$\sim (\sim p) \Leftrightarrow p$	Double negation
2.	$(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$	Contrapositive
3a.	$(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$	Implication
3b.	$(p \rightarrow q) \Leftrightarrow \sim (p \wedge \sim q)$	Implication
4.	$(p \leftrightarrow q) \Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$	Equivalence
5a.	$p \wedge p \Leftrightarrow p$	Idempotent law for conjunction
5b.	$p \vee p \Leftrightarrow p$	Idempotent law for disjunction
6a.	$(p \wedge q) \Leftrightarrow (q \wedge p)$	Commutative law for conjunction
6b.	$(p \vee q) \Leftrightarrow (q \vee p)$	Commutative law for disjunction
6c.	$(p \leftrightarrow q) \Leftrightarrow (q \leftrightarrow p)$	Commutative law for biconditional
7a.	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	Associative law for conjunction
7b.	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	Associative law for disjunction

8a.	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributive law for conjunction
8b.	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	Distributive law for disjunction
9a.	$\sim (p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$	De Morgan's law
9b.	$\sim (p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$	De Morgan's law
10a.	$p \vee (p \wedge q) \Leftrightarrow p$	Absorption law
10b.	$p \wedge (p \vee q) \Leftrightarrow p$	Absorption law
11a.	$p \vee \sim p \Leftrightarrow t$	Inverse law
11b.	$p \wedge \sim p \Leftrightarrow c$	Inverse law
12a.	$p \vee t \Leftrightarrow t$	Identity law
12b.	$p \wedge t \Leftrightarrow p$	Identity law
12c.	$p \vee c \Leftrightarrow p$	Identity law
12d.	$p \wedge c \Leftrightarrow c$	Identity law

Any one of the laws of logic can be verified by constructing truth tables. We illustrate this by verifying one of De Morgan's laws.

EXAMPLE 1   **De Morgan's First Law**   Establish De Morgan's first law by constructing a truth table

Solution   Construct the truth table as shown.

p	q	$p \vee q$	$\sim (p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim (p \wedge q) \leftrightarrow (\sim p \wedge \sim q)$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Since $\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$ is a tautology, $\sim (p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$.   ✦

EXAMPLE 2   **Using De Morgan's Laws**   Negate the statement: "The author's name is Mud, and Abraham Lincoln was president of the United States" using De Morgan's second law.

Solution   Let p and q be the statements:

p: The author's name is Mud.
q: Abraham Lincoln was a president of the United States.

Then the given statement is $p \wedge q$. The negation is $\sim (p \wedge q)$, which is logically equivalent to $\sim p \vee \sim q$ according to De Morgan's first law. This can be written as "The author's name is not Mud, or Abraham Lincoln was not president of the United States."   ✦

Just as one can give a proof of a theorem based on the validity of other theorems, so also one can establish a law of logic based on the validity of other laws of logic. The following is an example.

EXAMPLE 3   **Establishing Additional Laws of Logic**   Show that

$$p \vee (q \vee r) \Leftrightarrow r \vee (q \vee p)$$

Solution

$$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r \text{ Law 7b}$$
$$\Leftrightarrow r \vee (p \vee q) \text{ Law 6b}$$
$$\Leftrightarrow r \vee (q \vee p) \text{ Law 6b}$$

♦

In algebra a complex expression can sometimes be simplified. For example, the expression $-(x-a)-a$ is the same as $-x$. In a similar way in logic a complex statement can sometimes be simplified, that is, written in a logically equivalente simpler statement. The following is an example,

EXAMPLE 4   **Simplifying a Logical Statement**   Simplify

$$\sim (p \to q) \vee (p \wedge q)$$

Solution

$\sim (p \to q) \vee (p \wedge q)$	$\Leftrightarrow$	$\sim (\sim p \vee q) \vee (p \wedge q)$	Implication
	$\Leftrightarrow$	$[(\sim\sim p) \wedge \sim q] \vee (p \wedge q)$	De Morgan's law
	$\Leftrightarrow$	$(p \wedge \sim q) \vee (p \wedge q)$	Double negative
	$\Leftrightarrow$	$p \wedge (\sim q \vee q)$	8a
	$\Leftrightarrow$	$p \wedge (q \vee \sim q)$	6b
	$\Leftrightarrow$	$p \wedge t$	11a
	$\Leftrightarrow$	p	12b

Thus the rather complicated expression $\sim (p \to q) \vee (p \wedge q)$ is logically equivalent to p.

♦

✧ Implication

In the next section we will be making a logical argument. Certainly replacing a statement with its logical equivalence is very useful. However, what we really will need to do is replace a statement with one that is logically implied by it. We have the following definition:

> **Logical Implication**
>
> We say that the statement P logically implies the statement Q, denoted by
>
> $$P \Rightarrow Q$$
>
> if Q is true whenever P is true.

We use capital letters P and Q to emphasize that P and Q are usually compound statements.

EXAMPLE 5   **Verifying a Logical Implication**   Show $\sim q \wedge (p \to q) \Rightarrow \sim p$.

Solution   The truth table for $[\sim q \wedge (p \to q)]$ (denoted also by P) and $\sim p$ (denoted also by Q) is

					P	Q
p	q	$\sim q$	$p \to q$	$\sim q \wedge (p \to q)$		$\sim p$
T	T	F	T	F		F
T	F	T	F	F		F
F	T	F	T	F		T
F	F	T	T	T		T

Notice from the table that Q is true whenever P is true. Thus $P \Rightarrow Q$ or $\sim q \wedge (p \to q) \Rightarrow \sim p$. ✦

The truth table for the conditional $P \to Q$ is reproduced in Table L.18. Notice that $P \to Q$ will be a tautology if the second case never arises. But the second case never arises precisely when Q is true whenever P is. That is, precisely when $P \Rightarrow Q$. We have then established the following equivalent definition of logical equivalence in Table L.18.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table L.18

> **Logical Implication**
> We say that the statement P logically implies the statement Q, denoted by
> $$P \Rightarrow Q$$
> if $P \to Q$ is a tautology.

We suggest that one use this latter definition to verify logical implications.

EXAMPLE 6   **Verifying a Logical Implication**   Show

$$(p \to q) \wedge (q \to r) \Rightarrow p \to r$$

Solution   First construct the truth table for $[(p \to q) \wedge (q \to r)] \to (p \to r)$.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The table indicates that $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology. Thus $(p \to q) \wedge (q \to r) \Rightarrow p \to r$. ✦

The following list gives some additional logical implications.

Logical Implications

1. $p \wedge (p \to q) \Rightarrow q$

2. $\sim p \wedge (p \vee q) \Rightarrow q$

3. $p \wedge p \Rightarrow p$

4. $p \Rightarrow p \vee q$

5. $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$

6. $(p \wedge q) \rightarrow r \Rightarrow p \rightarrow (q \rightarrow r)$

7. $p \rightarrow (q \rightarrow r) \Rightarrow (p \wedge q) \rightarrow r$

8. $p \rightarrow (q \wedge \sim q) \Rightarrow \sim p$

9. $(p \leftrightarrow q) \wedge (q \leftrightarrow r) \Rightarrow p \leftrightarrow r$

10a. $(p \rightarrow q) \vee (r \rightarrow s) \Rightarrow (p \vee r) \rightarrow (q \vee s)$

10b. $(p \rightarrow q) \wedge (r \rightarrow s) \Rightarrow (p \wedge r) \rightarrow (q \wedge s)$

Self-Help Exercises L.4

1. Prove that $\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$.

2. Prove that $p \wedge (p \rightarrow q) \Rightarrow q$.

L.4 Exercises

Establish the laws of logic that are given in Exercises 1 through 14, by using truth tables. The statement t is a tautology and the statement c is a contradiction.

1. $p \wedge p \Rightarrow p$

2. $p \vee p \Rightarrow p$

3. $p \vee t \Rightarrow t$

4. $p \wedge t \Rightarrow p$

5. $p \wedge c \Rightarrow c$

6. $p \vee c \Rightarrow p$

7. $(p \wedge q) \wedge r \Rightarrow p \wedge (q \wedge r)$

8. $(p \vee q) \vee r \Rightarrow p \vee (q \vee r)$

9. $\sim p \wedge (p \vee q) \Rightarrow q$

10. $p \Rightarrow q \rightarrow (p \wedge q)$

11. $(p \leftrightarrow q) \wedge (q \leftrightarrow r) \Rightarrow p \leftrightarrow r$

12. $(p \wedge q) \rightarrow r \Rightarrow p \rightarrow (q \rightarrow r)$

13. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

14. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Establish the laws of logic given in Exercises 15 through 20.

15. $(p \vee q) \vee \sim q \Leftrightarrow p$

16. $p \vee (\sim p \wedge \sim q) \Leftrightarrow p \vee \sim q$

17. $p \wedge q \Leftrightarrow \sim (\sim p \vee \sim q)$

18. $p \vee q \Leftrightarrow \sim (\sim p \wedge \sim q)$

19. $(\sim p \wedge \sim q) \wedge \sim r \Leftrightarrow [p \vee (q \vee r)]$

20. $\sim [p \vee (q \vee r)] \Leftrightarrow (\sim p \wedge \sim q) \wedge \sim r$

In Exercises 21 through 24, rewrite each using only the connectives $\sim$ and $\vee$.

21. $p \wedge q$ **22.** $\sim p \wedge \sim q$

23. $p \rightarrow q$ **24.** $p \leftrightarrow q$

In Exercises 25 through 28, rewrite using only the connectives $\sim$ and $\wedge$.

25. $p \vee q$ **26.** $\sim p \vee \sim q$

27. $p \rightarrow q$ **28.** $p \leftrightarrow q$

In Exercises 29 through 32, negate the statements using the De Morgan laws.

29. Jim likes Sue or Mary.

30. Jim likes Sue and Mary.

31. Sales are up and we all get raises.

32. Sales are up or we all get raises.

33. Show that $\sim (p \rightarrow q) \Leftrightarrow p \wedge \sim q$.

In Exercises 34 through 37, use the result of Exercise 33 to negate the given statement.

34. If I do all my homework, then I will pass this course.

35. If I work hard, then I will succeed.

36. If interest rates go down, then stock prices will go up.

37. If prices are raised, then sales will drop.

Solutions to Self-Help Exercises L.4

1. The truth table for $[\sim (p \wedge q)] \leftrightarrow [\sim p \wedge \sim q]$ indicates that this biconditional is a tautology. Thus $\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

2. The truth table for $[p \wedge (p \rightarrow q)] \rightarrow q$ indicates that this conditional is a tautology. Thus, $p \wedge (p \rightarrow q) \Rightarrow q$.

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

L.5   Arguments

✦ Valid Arguments

Consider the following argument:

> If interest rates are falling, then stock prices are rising.
> Interest rates are falling.
> Therefore, stock prices are rising.

We refer to the first two statements as *hypothesis* and to the third statement as the *conclusion*. We do not wish to concern ourselves with whether any of these statements are true. Rather, we ask if the truth of the conclusion follows logically from the assumed truth of the hypothesis.

Let p be the statement "Interest rates are falling" and q the statement "Stock prices are rising." It is convenient to write the above argument in symbolic form as

$$p \rightarrow q$$
$$p$$
$$\therefore q$$

where the symbol $\therefore$ means "therefore." Then we ask if q is true whenever both $p \rightarrow q$ and p are true. That is, we ask if q is true whenever $(p \rightarrow q) \wedge p$ is true. But this happens precisely when $[(p \rightarrow q) \wedge p] \rightarrow q$ is a tautology, since $[(p \rightarrow q) \wedge p] \rightarrow q$ is a tautology precisely when the case $(p \rightarrow q) \wedge p$ is true while q is false never occurs.

The truth table, Table L.19, indicates that the statement $[(p \rightarrow q) \wedge p] \rightarrow q$ is a tautology. Recall from the last section that we denote this by $[(p \rightarrow q) \wedge p] \Rightarrow q$. In general we have the following:

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Table L.19

Argument, Hypothesis, Conclusion
An **argument** consists of a set of statements h_1, h_2, ... , h_n, called **hypotheses** and a statement q called the **conclusion**. An argument is **valid** if the conclusion is true whenever the hypotheses are all true, that is, whenever

$$h_1 \wedge h_2 \wedge \cdots h_n \Rightarrow q$$

It is very important to understand the distinction between the two words "valid" and "true." Consider the following argument

If George Washington was a man, then he was never president of the United States.

George Washington was a man.

Therefore, George Washington was never president of the United States.

Let p be "George Washington was a man" and q be "George Washington was never president of the United States." Then this argument can be written symbolically as

$$p \rightarrow q$$
$$\underline{p \qquad\qquad}$$
$$\therefore q$$

As we saw above, this is a valid argument. The conclusion, however, we know to be false. Perhaps this can be characterized as garbage-in, garbage-out. It is, of course, always possible that, despite garbage-in, the output is good. Consider the following argument:

If George Washington was not a man, then he was president of the United States.

George Washington was not a man.

Therefore, George Washington was president of the United States.

If p is "George Washington was not a man" and q is "George Washington was president of the United States," then this argument can also be written symbolically as the previous two were, and thus is a valid argument. However, one of the hypotheses is false while the conclusion is true.

EXAMPLE 1   Determining if an Argument Is Valid   Determine if the following argument is valid.

If interest rates are falling, then stock prices are rising.

Stock prices are rising.

Therefore, interest rates are falling.

Solution   Let p be the statement "Interest rates are falling" and q the statement "Stock prices are rising." Then this argument can be written as

$$p \rightarrow q$$
$$\underline{q \qquad\qquad}$$
$$\therefore p$$

Thus, we wish to determine if $[(p \rightarrow q) \wedge q] \rightarrow p$ is a tautology. Create a truth table.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge q] \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

The truth table indicates that this is not a tautology and therefore the given argument is not valid.   ✦

EXAMPLE 2   Determining if an Argument Is Valid   Determine if the following argument is valid.

Jim takes an economics course or Jim takes a math course.
<u>Jim does not take an economics course.</u>
Therefore, Jim takes a math course.

Solution   Let p be the statement "Jim takes an economics course" and q be the statement "Jim takes a math course." Then this argument can be written as

$p \wedge q$
<u>$\sim p$</u>
$\therefore q$

Thus, we wish to determine if $[(p \vee q) \wedge \sim p] \rightarrow q$ is a tautology. Create a truth table.

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim p$	$[(p \vee q) \wedge \sim p] \rightarrow q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

The truth table indicates that this statement is a tautology and so the argument is valid.   ✦

✧ Using Implications and Equivalences in Arguments

Using truth tables to verify that an argument is valid is a mechanical process. Even if there are a large number of hypotheses, a computer should be able to handle the construction of the truth table and the verification of the appropriate tautology. But this is not enough. Mechanically constructing a truth table does not lead to insight into the argument, or to possible simplifications, or to generalizations. To obtain this greater insight and to get a better feel for logical arguments in mathematics in general, we need to construct proofs based on the logical inferences and equivalences that were established in the last section.

In the following problem, instead of creating a truth table, we establish a proof by using some of the logical implications and equivalences established in the last section. In the following list we present some of these again.

1. $p \Rightarrow p \vee q$ Addition
2. $p \wedge q \Rightarrow p$ Subtraction
3. $p \wedge (p \rightarrow q) \Rightarrow q$ Modus ponens
4. $(p \rightarrow q) \wedge \sim q \Rightarrow \sim p$ Modus tollens
5. $(p \vee q) \wedge \sim p \Rightarrow q$ disjunctive syllogism
6. $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$ Hypothetical syllogism

EXAMPLE 3   Constructing a Proof   Suppose the following are true:

Janet is healthy or she is not wealthy.
If Janet is healthy, then she plays tennis.

Janet does not play tennis.

Prove that Janet is not wealthy.

Solution   Let h, w, and p be the following statements:

h: Janet is healthy.
w: Janet is wealthy.
p: Janet plays tennis.

Then we wish to establish the following:

$h \lor \sim w$
$h \rightarrow p$
$\therefore \sim w$

We can establish this without using a truth table as follows:

$(h \lor \sim w) \land (h \rightarrow p) \land (\sim p) \Rightarrow (h \lor \sim w) \land (\sim h)$ Modus tollens
$\Rightarrow \sim w$ Disjunctive syllogism

Thus $\sim w$ is true, and this means Janet is not wealthy.   ✦

EXAMPLE 4   **Constructing a Proof**   Suppose the following are true:

Joe goes to the mountains, or he goes to the beach.
If Joe does not take his convertible, then he does not go to the beach.
Joe does not go to the mountains.

Prove that Joe takes his convertible.

Solution   Let m, b, and h be the following statements:

m: Joe goes to the mountains.
b: Joe goes to the beach.
h: Joe takes his convertible.

Then we wish to prove the following:

$m \lor b$
$\sim h \rightarrow \sim b$
$\sim m$
$\therefore \sim w$

This can be established without using truth table as follows:

$(m \lor b) \land (\sim h \rightarrow \sim b) \land (\sim m) \Leftrightarrow (m \lor b) \land (\sim m) \land (\sim h \rightarrow \sim b)$ Commutative law
$\Rightarrow b \land (\sim h \rightarrow \sim b)$ Disjunctive syllogism
$\Leftrightarrow b \land (b \rightarrow h)$ Contrapositive
$\Rightarrow h$ Modus ponens

Thus h is true, and this means Joe takes his convertible.   ✦

EXAMPLE 5   **Constructing a Proof**   Suppose the following are true:

If Joe does not play tennis, then Joe does not drive to the mountains.
If Joe drives to the beach, then Joe does not rent a car.
If it is sunny, then Joe rents a car.
If Joe does not drive to the mountains, then Joe drives to the beach.
It is sunny.

Prove that Joe plays tennis.

Solution   Let p, m, b, r, and s be the following statements:

p: Joe plays tennis.
m: Joe drives to the mountains.
b: Joe drives to the beach.
r: Joe rents a car.
s: It is sunny.

The first five statements are then written symbolically as follows:

1. h_1: $\sim p \to \sim m$ Logically equivalent to $m \to p$
2. h_2: $b \to \sim r$ Logical equivalent to $r \to \sim b$
3. h_3: $s \to r$
4. h_4: $\sim m \to b$ Logically equivalent to $\sim b \to m$
5. h_5: s

The given logical equivalence for h_1 is just the contrapositive. The given logical equivalences for h_2 and h_4 use the contrapositive and the double negation. For example, the contrapositive for h_2 is $[\sim (\sim c)] \to \sim b$. Now, using the double negation, this becomes logically equivalent to $c \to \sim b$. Then

$$h_1 \wedge h_2 \wedge h_3 \wedge h_4 \wedge h_5$$
$$\Leftrightarrow (\sim p \to \sim m) \wedge (b \to \sim r) \wedge (s \to r) \wedge (\sim m \to b) \wedge s$$
$$\Leftrightarrow (m \to p) \wedge (r \to \sim b) \wedge (s \to r) \wedge (\sim b \to m) \wedge s$$

Now, using the commutative laws, we can write

$$h_1 \wedge h_2 \wedge h_3 \wedge h_4 \wedge h_5$$
$$\Leftrightarrow h_5 \wedge h_3 \wedge h_2 \wedge h_4 \wedge h_1$$
$$\Leftrightarrow s \wedge (s \to r) \wedge (r \to \sim b) \wedge \sim b \to m) \wedge (n \to p)$$

Now making repeated use of modus ponens yields

$$p \wedge (p \to r) \wedge (r \to \sim b) \wedge (\sim b \to m) \wedge (m \to p)$$
$$\Rightarrow r \wedge (r \to \sim b) \wedge (\sim b \to m) \wedge (m \to p)$$
$$\Rightarrow (\sim b) \wedge (\sim b \to m) \wedge (m \to p)$$
$$\Rightarrow m \wedge (m \to p)$$
$$\Rightarrow p$$

Thus p is true, and this means that Joe plays tennis.   ✦

✧ Indirect Proof

In order to establish

$$h_1 \wedge h_2 \wedge \cdots \wedge h_n \Rightarrow q$$

it is sometimes easier to establish the contrapositive

$$\sim q \Rightarrow \sim (h_1 \wedge h_2 \wedge \cdots \wedge h_n)$$

Using the De Morgan law, this can be written as

$$\sim q \Rightarrow (\sim h_1) \vee (\sim h_2) \vee \cdots \vee (\sim h_n)$$

To show this, we must assume that $\sim q$ is true and then show that at least one of the statements $\sim h_1, \sim h_2, \cdots, \sim h_n$, is true. That is, we must assume that q is false and show that one of the statements $h_1, h_2, \ldots, h_n$, is false.

Indirect Proof

In order to give an indirect proof of

$$h_1 \wedge h_2 \wedge \cdots \wedge h_n \Rightarrow q$$

we assume that the conclusion q is false and then prove that at least one of the hypotheses $h_1, h_2, \ldots, h_n$, is false.

EXAMPLE 6   Indirect Proof   Give an indirect proof for Example 5.

Solution   For convenience the hypotheses are listed again.

1. $h_1: \sim p \rightarrow \sim m$
2. $h_2: b \rightarrow \sim r$
3. $h_3: s \rightarrow r$
4. $h_4: \sim m \rightarrow b$
5. $h_5: s$

The conclusion is the statement p. For an indirect proof we assume that p is false, that is, that $\sim p$ is true. Naturally if h_1, h_2, h_3, or h_4 are false, we are finished. We will complete the proof by assuming that h_1, h_2, h_3, and h_4 are true, and then show that h_5 must be false. Then h_1 implies that $\sim m$ is true, and then h_4 implies that b is true, and then h_2 implies $\sim r$ is true. The contrapositive of h_3 is $\sim r \rightarrow \sim s$, and thus since $\sim r$ is true this indicates that $\sim s$ is true. But this states that the fifth hypothesis, h_5, is false. Thus one of the hypotheses has been shown to be false given that the conclusion is false. This then establishes the required results by an indirect proof.   ✦

Self-Help Exercises L.5

1. Establish the following argument using truth tables.

$$p \wedge q$$
$$\underline{\sim p \qquad\quad}$$
$$\therefore q$$

2. Establish the following without using truth tables.

If it is raining, then I will not be at the beach. If I have money for gas for my car, then I will be at the beach. I have money for gas for my car. Therefore, it is not raining.

3. Use an indirect proof to establish the argument in the previous exercise.

L.5 Exercises

In Exercises 1 through 10, use truth tables to establish whether or not the arguments are valid.

1. $p \vee q$
$\dfrac{\sim p}{\therefore q}$

2. $p \rightarrow q$
$\dfrac{\sim q}{\therefore \sim p}$

3. $p \rightarrow q$
$q \rightarrow r$
$\dfrac{}{\therefore p \rightarrow r}$

4. $p \rightarrow q$
$q \vee r$
$\dfrac{}{\therefore p \vee r}$

5. $p \leftrightarrow q$
q
$\dfrac{}{\therefore p}$

6. $p \rightarrow q$
$q \rightarrow p$
$\dfrac{}{\therefore p \leftrightarrow q}$

7. $p \rightarrow \sim q$
$r \rightarrow \sim q$
r
$\dfrac{}{\therefore \sim p}$

8. $(\sim p \rightarrow \sim q)$
$(r \rightarrow \sim p)$
q
$\dfrac{}{\therefore \sim r}$

9. $p \rightarrow q$
$p \rightarrow r$
$\dfrac{}{\therefore q \wedge r}$

10. $p \rightarrow q$
$\sim q \rightarrow \sim r$
$\dfrac{}{\therefore p \rightarrow \sim r}$

In Exercises 11 through 14, without using truth tables, determine if the arguments are valid.

11. If I go out with my sister or my friend, then I do not drive my car. I am driving my car. Therefore, I am not out with my sister.

12. If it is Saturday, then I sleep in. Today is Saturday or not Sunday. Today is Sunday. Therefore, I sleep in.

13. You are happy if and only if you are healthy. You are healthy or you are smart. You are not happy. Therefore, if you are not happy, then you are not smart.

14. If it is snowing, then I go skiing. If I go skiing, then I am happy. I am happy. Therefore it is snowing.

In Exercises 15 through 26, show that the argument is valid without using truth tables.

15. If I have an egg and orange juice for breakfast, then I do not have cereal. If I do not have orange juice, then I do not have grapefruit juice. I had an egg and grapefruit juice this morning for breakfast. Therefore, I did not have cereal.

16. If I study hard, then I make good grades and am not depressed. I am not depressed. I did not make good grades. Therefore, I did not study hard.

17. If I clean my room, my mother is not mad. My mother is mad. Therefore, I did not clean my room.

18. If the price of lumber rises, then the price of houses rises. If the price of steel rises, then the price of cars rises. The price of lumber is rising or the price of steel is rising. Therefore, the price of houses is rising or the price of cars is rising.

19. Jane eats broccoli and spinach. If Jane eats broccoli, then she does not eat spinach. Therefore Jane eats spinach and does not eat broccoli.

20. I eat an apple or an orange every day. If I do not eat lunch, then I do not eat an apple. I did not eat lunch today. Therefore, I ate an orange.

21. If I do not play baseball, then I play soccer. If I do not play soccer, then I play basketball. Therefore, if I do not play soccer, then I play baseball or basketball.

22. If I work hard or am smart, then I will not fail. If I do not work hard, then I fail and am poor. I failed. Therefore, I am poor.

23. If the rate of inflation increases, then the price of gold increases. If the price of gold increases, then the prices of bonds do not increase. The prices of bonds are increasing. Therefore, the rate of inflation is not increasing.

24. If the summer is hot and dry, then the crops will be poor. If the summer is dry, then there are no floods. Therefore, if there are floods and poor crops, then the summer is not hot.

25. If the price of sugar rises, then the price of candy rises and I eat less candy. The price of candy is not rising. I am not eating less candy. Therefore, the price of sugar is not rising.

26. If I go out Friday night and Saturday night, then I sleep in on Sunday. I always go out on Saturday

night. Therefore, if I go out on Friday night, I will sleep in on Sunday.

27. Use an indirect proof to establish the argument in Exercise 11.

28. Use an indirect proof to establish the argument in Exercise 20.

29. Use an indirect proof to establish the argument in Exercise 23.

30. Use an indirect proof to establish the argument in Exercise 25.

Solutions to Self-Help Exercises L.5

1. The following truth table indicates that $[(p \wedge q) \wedge (\sim p)] \to q$ is a tautology. Thus the argument is valid.

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \wedge (\sim p)$	$[(p \wedge q) \wedge (\sim p)] \to q$
T	T	T	F	F	T
T	F	F	F	F	T
F	T	F	T	F	T
F	F	F	T	F	T

2. Let p, q, and r be the following statements.

r: It is raining.

b: I am at the beach.

g: I have money for gasoline for my car.

Then the argument can be written as follows.

$r \to \sim b$
$g \to b$
g
_____
$\therefore \sim r$

The proof can be given as follows.

$$(r \to \sim b) \wedge (g) \wedge (g \to b) \quad \Rightarrow \quad (r \to \sim b) \wedge (b) \quad \text{Modus ponens}$$
$$\Rightarrow \quad \sim r \quad \text{Modus tollens}$$

The following is another proof.

$$(g) \wedge (g \to b) \wedge (r \to \sim b) \quad \Rightarrow \quad (b) \wedge (r \to \sim b) \quad \text{Modus ponens}$$
$$\Rightarrow \quad (b) \wedge (b \to \sim r) \quad \text{Contrapositive}$$
$$\Rightarrow \quad \sim r \quad \text{Modus ponens}$$

3. We begin the indirect proof by assuming the conclusion $\sim r$ is false. This means that r is true. If $r \to \sim b$ is true, we then have that $\sim b$ is true. If $g \to b$ is true, we then have $\sim g$ is true. But this contradicts the third hypothesis g.

L.6   Switching Networks

We will now see how the principles of logic can be used in the design and analysis of switching networks. A **switching network** consists of an energy input, such as a battery, an output, such as a light bulb, and an arrangement of wires and switches connecting the input and output. A **switch** is a device that is either **closed** or **open**. If the switch is closed, current will flow through the wire. If the switch is open, current will not flow. Because a switch has two states, we can represent the switch by a proposition p that is true if the switch is closed and false if the switch is open.
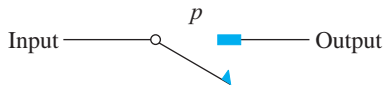
Normally a house switch on a wall can move in a vertical direction. Electricians usually wire such a switch so that the up position turns the light on (the switch is closed) and the down position turns the light off (the switch is open). Figure L.1 then indicates our schematic for a switch. Up indicates the switch is closed (the proposition p is true) and down indicates the switch is open (the proposition p is false).

Figure L.1

Consider now a network of two switches arranged as shown in Figure L.2 The switches p and q are said to be in **series**. For this network, current will flow from the input to the output if and only if both of the switches p and q are closed.

Figure L.2

Note that if one or more of the switches are open, then no current can flow. If we think of p and q as propositions and T corresponds to a closed switch and F to an open switch, then we can write the following truth table, Table L.20. We notice that this is the truth table for $p \wedge q$. Therefore, two switches p and q connected in series corresponds to the conjunction $p \wedge q$.

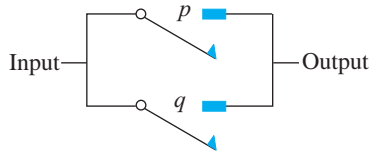| p	q	Current Flow?	Truth Assignment
T	T	yes	T
T	F	no	F
F	T	no	F
F	F	no	F

Table L.20

Figure L.3

Consider now a network of two switches arranged as shown in Figure L.3. The switches p and q are said to be in **parallel**. For this network, current will flow from the input to the output if, and only if, at least one of the switches p or q is closed. If we again think of p and q as propositions and T corresponds to a closed switch and F to an open switch, then we can write the following truth table, Table L.21. We notice that this is the truth table for $p \vee q$. Therefore, two switches p and q connected in parallel correspond to the (inclusive) disjunction $p \vee q$.

We will use the standard convention that the input is on the left and the output is on the right and will drop the words "input" and "output" from the figures.
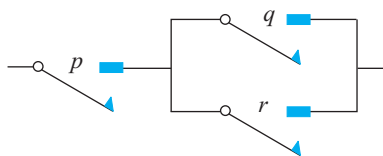
p	q	Current Flow?	Truth Assignment
T	T	yes	T
T	F	yes	T
F	T	yes	T
F	F	no	F

Table L.21

Figure L.4

EXAMPLE 1   **Using Truth Tables to Evaluate Networks**   Find a compound statement that represents the network shown in Figure L.4.By constructing a truth table for this compound statement, determine the conditions under which current will flow.

Solution   The compound statement is $p \wedge (q \vee r)$. Create the truth table.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

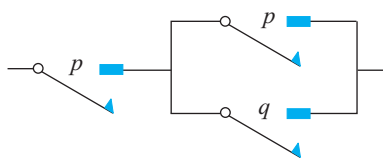From the truth table we see that current will flow whenever p is closed and either q or r (or both) are closed. ✦

Two different switches can be wired so that they open and close simultaneously. In such a case, we let the same letter designate each switch. Consider the network shown in Figure L.5. By examining the network in Figure L.5 we see that when the switch p is closed, current flows. When the switch is open, current does not flow. Thus, this network is equivalent to the one shown in Figure L.1. We can also use the laws of logic to show this.

Figure L.5

EXAMPLE 2   **Using Truth Tables to Evaluate Networks**   Find a compound statement that represents the network shown in Figure L.5. By constructing a truth table for this compound statement, determine a simpler network that is equivalent.

Solution   The network in Figure L.5 corresponds to the compound statement $p \wedge (p \vee q)$. Create a truth table.

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Since the compound statement $p \wedge (p \vee q)$ is equivalent to p, the network in Figure L.5 is equivalent to the one in Figure L.1.

✦

The previous two examples indicate how a network can be viewed as a compound statement. The last example indicates how the known laws of logic can sometimes be used to determine an equivalent and simpler network. This type of simplification is extremely important in the economical design of networks for practical problems.

Suppose we have a switch p. A second switch can be wired so that the second switch will be closed when p is open and open when p is closed. In such a case we designate the second switch by $\sim p$ since the statement $\sim p$ is true when p is false and $\sim p$ is false when p is true.

EXAMPLE 3   **Determining Equivalent Networks**   Find an equivalent and simpler network for each of the networks shown in Figure L.6.

Figure L.6a
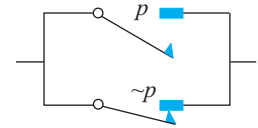
Figure L.6b

Solution

a. This network corresponds to the compound statement $p \wedge \sim p$, which is always false. An equivalent network is shown in Figure L.7a. This is a network with a permanent break and current never flows.

Figure L.7a   **Figure L.7b**

b. This network corresponds to the compound statement $p \vee \sim p$ which is always true. An equivalent network is shown in Figure L.7b. This is a network in which current always flows.   ✦

EXAMPLE 4   **Simplifying a Network**   Simplify the network shown in Figure L.8.
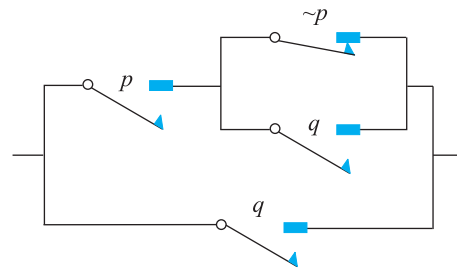
Figure L.8

Solution   The corresponding compound statement is $[p \wedge (\sim p \vee q)] \vee q$. Using the rules of logic, we then have

$$[p \wedge (\sim p \vee q)] \vee q \Leftrightarrow [(p \wedge \sim p) \vee (p \wedge q)] \vee q \quad \text{Distributive law}$$
$$\Leftrightarrow [c \vee (p \wedge q)] \vee q \qquad\qquad c \text{ a contradiction}$$
$$\Leftrightarrow (p \wedge q) \vee q \qquad\qquad\quad \text{Identity law}$$
$$\Leftrightarrow q \qquad\qquad\qquad\qquad\quad \text{Absorption law}$$

Thus this network can be replaced with the equivalent network shown in Figure L.1.   ✦

EXAMPLE 5   **Simplifying a Network**   Simplify the network shown in Figure L.9.
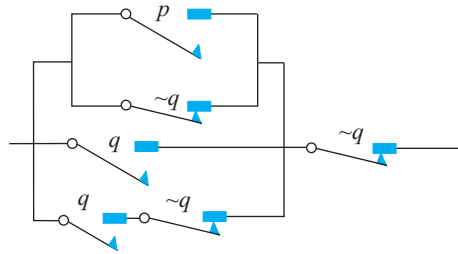
Figure L.9

Solution   The corresponding compound statement is $(\sim q) \wedge [(p \vee \sim q) \vee q \vee (q \wedge \sim q)]$. Using the rules of logic we then have.

$$(\sim q) \wedge [(p \vee \sim q) \vee q \vee (q \wedge \sim q)]$$

$$\Leftrightarrow \quad (\sim q) \wedge [p \vee ((\sim q) \vee q) \wedge (q \wedge \sim q)] \quad \text{Associative law}$$
$$\Leftrightarrow \quad (-q) \wedge [(p \vee t) \vee (q \wedge \sim q)] \qquad\quad \text{Inverse law}$$
$$\Leftrightarrow \quad (\sim q) \wedge [t \vee (q \wedge \sim q)] \qquad\qquad\quad \text{Identity law}$$
$$\Leftrightarrow \quad (\sim q) \wedge t \qquad\qquad\qquad\qquad\qquad \text{Identity law}$$
$$\Leftrightarrow \quad \sim q \qquad\qquad\qquad\qquad\qquad\qquad \text{Identity}$$
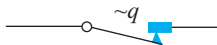
Figure L.10

Hence the compound statement is equivalent to the statement $\sim q$. Thus, this network can be replaced with the equivalent network shown in Figure L.10.   ✦

Self-Help Exercises L.6

1. Find a logic statement that corresponds to the following network.
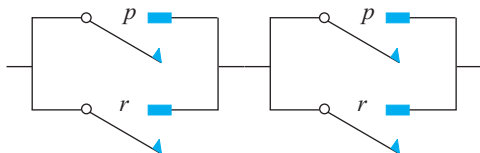
2. Simplify the statement in the previous exercise by using the laws of logic. Then draw a network that corresponds to the simplified statement.

L.6 Exercises

In Exercises 1 through 4, find a logic statement corresponding to the given network. Determine conditions under which current will flow.

1.

2.

3.

4.
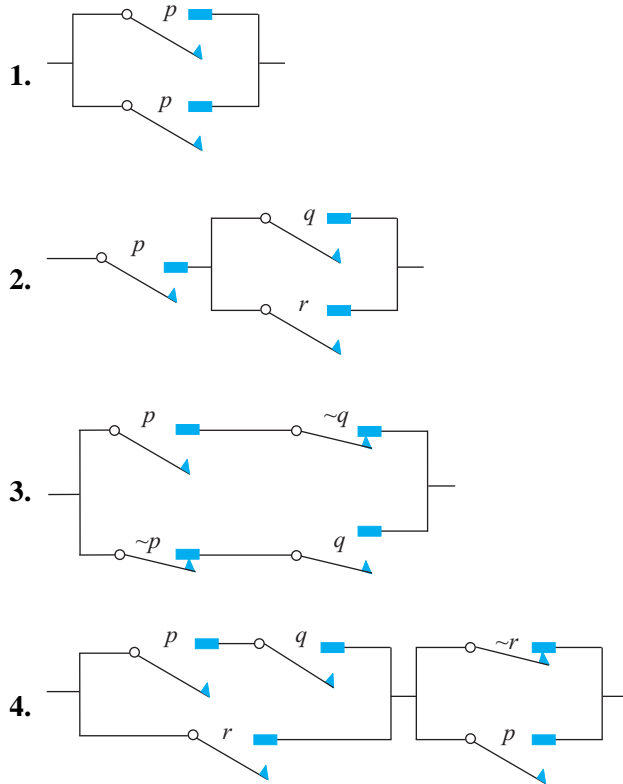
In Exercises 5 through 8, draw the network corresponding to the given logic statement.

5. $p \vee (q \wedge r)$

6. $(p \wedge q) \wedge r$

7. $(p \vee \sim q) \vee (q \wedge r)$

8. $[p \vee (\sim q \wedge r)] \wedge q$

In Exercises 9 through 16, find a logic statement corresponding to the given network. Then find a simpler but equivalent network.

9.

10.

11.

12.

13.

14.

15.

16.
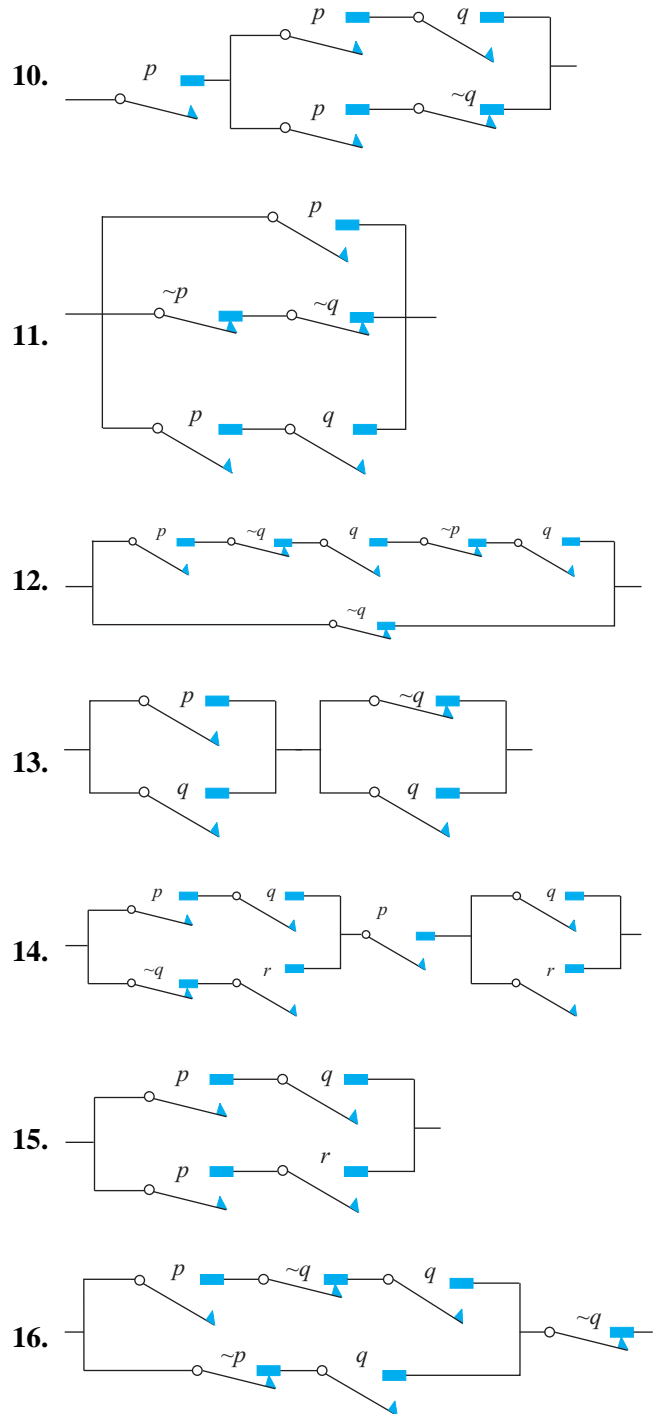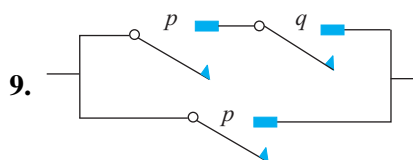
17. A room has two doors with a light switch near each door. Design a network so that each switch can turn the light on and turn it off.

18. A large room has three different switches. Design a network so that any one switch can turn the lights on and off.

Solutions to Self-Help Exercises L.6

1. The network corresponds to the statement $(p \vee r) \wedge (q \vee r)$.

2. Using the distributive law we have

$$(p \vee r) \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee r$$

The following network corresponds to this simplified statement.
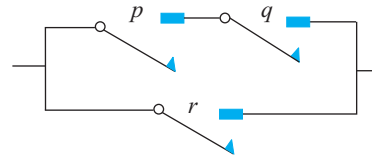
Review

✧ **Summary Outline**

- A **statement** is a declarative sentence that is either true or false but not both.

- A **connective** is a word or words, such as "and" or "if and only if," that is used to combine two or more simple statements into a compound statement.

- A **conjunction** is a statement of the form "p and q" and is written symbolically as $p \wedge q$. The conjunction $p \wedge q$ is true if both p and q are true, otherwise it is false.

- A **disjunction** is a statement of the form "p or q" and is written symbolically as $p \vee q$. The disjunction $p \vee q$ is false if both p and q are false and is true in all other cases.

- A **negation** is a statement of the form "not p" and is written symbolically as $\sim p$. The negation $\sim p$ is true if p is false and false if p is true.

- **Truth tables of some important connectives**

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	$\sim p$
T	F
F	T

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- We say that a statement is a **contradiction** if the truth value of the statement is always false no matter what the truth values of its simple component statements.

- We say that a statement is a **tautology** if the truth value of the statement is always true no matter what the truth values of its simple component statements.

- A **conditional statement** is a compound statement of the form "if p, then q" and is written symbolically as $p \rightarrow q$. A conditional statement is false if p is true and q is false, and is true in all other cases.

- In the conditional statement $p \rightarrow q$, the statement p is the **hypothesis** and the statement q is the **conclusion**.

- A **biconditional statement** is a compound statement of the form "p if and only if q" and is written symbolically as $p \leftrightarrow q$. The biconditional $p \leftrightarrow q$ is true only when p and q are both true or both false.

- Two statements p and q are **logically equivalent**, denoted by $p \Leftrightarrow q$, if they have identical truth tables.

- The **contrapositive** of the statement "if p, then q" ($p \rightarrow q$) is of the form "if not q, then not p" and is written $\sim q \rightarrow \sim p$.

- The **converse** of the conditional statement "if p, then q" ($p \rightarrow q$) is "if q, then p" and is written $q \rightarrow p$.

- The **inverse** of the conditional statement "if p, then q" ($p \rightarrow q$) is "if not p, then not q" and is written $\sim p \rightarrow \sim q$.

- Two statements p and q are **logically equivalent**, written $p \Leftrightarrow q$, if the biconditional $p \leftrightarrow q$ is a tautology.

- The **laws of logic** are found on pages 19–20.

- We say that the statement P **logically implies** the statement Q, denoted by $P \Rightarrow Q$, if Q is true whenever P is true. Another way of stating this is that $P \rightarrow Q$ is a tautology.

- An **argument** consists of a set of statements $h_1, h_2, \ldots, h_n$, called **hypotheses** and a statement q called the **conclusion**. An argument is **valid** if the conclusion is true whenever the hypotheses are all true, that is, whenever $h_1 \wedge h_2 \wedge \cdots \wedge h_n \Rightarrow q$.

- In order to give an **indirect proof** of $h_1 \wedge h_2 \wedge \cdots \wedge h_n \Rightarrow q$, we assume that the conclusion q is false and then prove that at least one of the hypotheses $h_1, h_2, \ldots, h_n$ is false.

- A **switching network** consists of an energy input, such as a battery, an output, such as a light bulb, and an arrangement of wires and switches connecting the input and output.

- A **switch** is a device that is either **closed** or **open**.

- Switches can be in **series** (see Figure L.2), or in **parallel** (see Figure L.3).

Review Exercises

In Exercises 1 through 4, decide which sentences are statements.

1. Water boils at 152^oF.

2. George Washington never told a lie.

3. Is it true that $2 + 2 = 4$?

4. $x^2 = -1$ has two solutions.

5. Let p and q denote the following statements:

  p: Abraham Lincoln was the shortest president of the United States.

  q: Carson City is the capital of Nevada.

Express the following compound statements in words.

   a. $\sim p$ **b.** $p \wedge q$ **c.** $p \vee q$

   d. $\sim p \wedge q$ **e.** $p \vee \sim q$ **f.** $\sim (p \wedge q)$

6. Let p and q denote the statements in the previous exercise.

   a. State the negation of these statements in words.

   b. State the disjunction for these statements in words.

   c. State the conjunction for these statements in words.

7. Let p and q denote the following statements:

   p: GM makes trucks.

   q: GM makes toys.

   Express the following statement symbolically

   a. GM does not make toys.

   b. GM makes trucks, or GM makes toys.

   c. GM makes trucks, or GM does not make toys.

   d. GM does not make trucks, and GM does not make toys.

In Exercises 8 and 9, construct truth tables for the given statements.

8. $(\sim p \wedge q) \wedge (\sim p \vee q)$

9. $(p \vee q) \vee (q \wedge \sim r)$

10. Determine if the statement

$$[(p \vee \sim q) \vee q \vee (\sim p \vee \sim q)] \vee \sim q$$

   is a tautology.

11. Determine if the statement

$$[(p \wedge \sim q) \wedge q] \vee (\sim p \wedge \sim q)$$

   is a contradiction.

12. Let p and q be the statements:

   p: China is in Asia.

   q: Mexico is in South America.

   Determine the truth value of the following statements.

   a. $\sim q$ **b.** $p \wedge \sim q$ **c.** $\sim p \wedge q$ **d.** $\sim (p \wedge q)$

13. Let p and q be the statements:

   p: Inflation is increasing.

   q: The price of gold is rising.

Write each of the following statements in English. Then write both in English and symbolically the contrapositive, inverse, and converse of the given statement.

   a. $p \rightarrow q$ **b.** $p \rightarrow \sim q$ **c.** $\sim q \rightarrow \sim p$

In Exercises 14 and 15, construct a truth table for the given statement.

14. $\sim (\sim p \rightarrow q)$

15. $(p \wedge q) \rightarrow r$

In Exercises 16 through 19, establish the given law of logic.

16. $p \wedge (p \rightarrow q) \Rightarrow q$

17. $p \rightarrow (q \rightarrow r) \Rightarrow (p \wedge q) \rightarrow r$

18. $p \vee (p \wedge q) \Leftrightarrow p$

19. $(p \rightarrow q) \Leftrightarrow \sim (p \wedge \sim q)$

20. Negate the following using the De Morgan laws.

   a. Sue will major in math or physics.

   b. Sue enjoys math and physics.

21. Use truth tables to establish whether or not the following argument is valid.

$$p \rightarrow \sim q$$
$$\sim r \rightarrow p$$
$$\underline{q}$$
$$\therefore r$$

22. Show that the following argument is valid without using truth tables.

   If it is sunny, then I do not study. If it is not sunny, then I am not at the beach. It is sunny and I am at the beach. Therefore, I am not studying.

23. Find a logic statement for the following network. Then find a simpler but equivalent network.
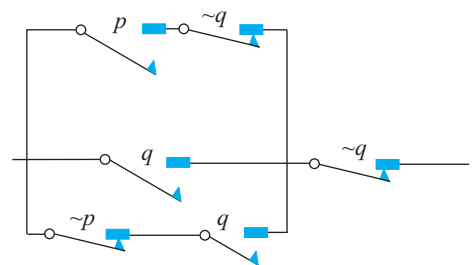
Answers to Selected Exercises

L.1 EXERCISES

1. Statement **3.** Statement **5.** Statement
7. Not a statement **9.** Statement
11. Not a statement **13.** Statement
15. a. George Washington was not the third president of the United States. **b.** George Washington was the third president of the United States, and Austin is the capital of Texas. **c.** George Washington was the third president of the United States, or Austin is the capital of Texas. **d.** George Washington was not the third president of the United States, and Austin is the capital of Texas. **e.** George Washington was the third president of the United States, or Austin is not the capital of Texas. **f.** It is not true that George Washington was the third president of the United States and that Austin is the capital of Texas.
17. a. George Washington did not own over $100,000$ acres of property. The Exxon Valdez was not a luxury liner. **b.** George Washington owned over $100,000$ acres of property, or the Exxon Valdez was a luxury liner.
c. George Washington owned over $100,000$ acres of property, and the Exxon Valdez was a luxury liner.
19. a. $\sim q$ **b.** $p \wedge \sim q$ **c.** $p \vee q$ **d.** $(\sim p) \vee (\sim q)$

L.2 EXERCISES

1.

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

3.

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

5.

p	q	$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \vee q$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	F

7.

p	q	$\sim p$	$p \wedge q$	$\sim p \vee (p \wedge q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

9.

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \wedge (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	F

11.

p	q	$\sim q$	$\sim p$	$p \vee \sim q$	$\sim p \wedge q$	$(p \vee \sim q) \vee (\sim p \wedge q)$
T	T	F	F	T	F	T
T	F	T	F	T	F	T
F	T	F	T	F	T	T
F	F	T	T	T	F	T

13.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

15.

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$\sim[(p \wedge q) \wedge r]$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

17.

p	q	r	$p \vee q$	$q \wedge r$	$(p \vee q) \vee (q \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	F	F	F

19.

p	q	r	~q	p∨~q	~q∧r	(p∨~q)∨(~q∧r)
T	T	T	F	T	F	T
T	T	F	F	T	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	T

21. a. True **b.** False **c.** True **d.** True **e.** False
23. a. True **b.** False **c.** False **d.** True

L.3 EXERCISES

1. If interest rates drop, then the stock market goes up. Contrapositive: If the stock market does not go up, then interest rates do not drop. $\sim q \to \sim p$ Inverse: If interest rates do not drop, then the stock market does not go up. $\sim p \to \sim q$ Converse: If the stock market goes up, then interest rates drop. $q \to p$

3. If the stock market does not go up, then interest rates do not drop. Contrapositive: If interest rates drop, then the stock market goes up. $p \to q$ Inverse: If the stock market goes up, then interest rates drop. $q \to p$ Converse: If interest rates do not drop, then the stock market does not go up. $\sim p \to \sim q$

5. If interest rates do not drop, then the stock market does not go up. $\sim p \to \sim q$ Contrapositive: If the stock market goes up, then interest rates drop. $q \to p$ Inverse: If interest rates drop, then the stock market goes up. $p \to q$ Converse: If the stock market does not go up, then interest rates do not drop. $\sim q \to \sim p$

7. $p \to q$ **9.** $p \to q$ **11.** $q \leftrightarrow p$ **13.** $\sim q \to \sim p$
15. True   **17.** False   **19.** True   **21.** False

23.

p	q	p→q	(p→q)∧p
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

25.

p	q	p→q	q→p	(p→q)∨(q→p)
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

27.

p	q	p∨q	~(p∨q)	~p	~q	~p∧~q	[~(p∨q)]↔[~p∧~q]
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

29.

p	q	r	p→q	(p→q)→r
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

31.

p	q	r	q∧r	p∨(q∧r)	p∨q	p∨r	(p∨q)∧(p∨r)	[p∨(q∧r)]↔[(p∨q)∧(p∨r)]
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	F	T

33.

p	q	p→q	q→p	(p→q)↔(q→p)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

35.

p	q	r	p∨q	(p∨q)∨r
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

37.

p	q	p∧q	(p∧q)→p
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Tautology

39.

p	q	p→q	p∧(p→q)	[p∧(p→q)]→q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Tautology

41.

p	q	$p \wedge q$	$q \to (p \wedge q)$	$p \to [q \to (p \wedge q)]$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

Tautology

43.

p	q	r	$p \to q$	$p \vee r$	$q \vee r$	$(p \vee r) \to (q \vee r)$	$[p \to q] \to [(p \vee r) \to (q \vee r)]$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	F	F	T	T

Tautology

45. Let p be the statement "n^2 is even" and q the statement "n is even." To prove $p \to q$, we will establish that $\sim q \to \sim p$. Assume $\sim q$, that is, assume that n is not even. Then n is not divisible by 2 and hence $nn = n^2$ is not divisible by 2. This means that n^2 is not even, that is, $\sim p$ is true.

L.4 EXERCISES

1.

p	$p \wedge p$	$(p \wedge p) \to p$
T	T	T
F	F	T

3.

p	t	$p \vee t$	$(p \vee t) \to t$
T	T	T	T
F	T	T	T

5.

p	c	$p \wedge c$	$(p \wedge c) \to c$
T	F	F	T
F	F	F	T

7.

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$	$[(p \wedge q) \wedge r] \to [p \wedge (q \wedge r)]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	F	T
T	F	F	F	F	F	F	T
F	T	T	F	F	T	F	T
F	T	F	F	F	F	F	T
F	F	T	F	F	F	F	T
F	F	F	F	F	F	F	T

9.

p	q	$\sim p$	$p \vee q$	$\sim p \wedge (p \vee q)$	$[\sim p \wedge (p \vee q)] \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

11.

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow r$	$(p \leftrightarrow q) \wedge (q \leftrightarrow r)$	$p \leftrightarrow r$	$[(p \leftrightarrow q) \wedge (q \leftrightarrow r)] \to [p \leftrightarrow r]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	F	T	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

13.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$ 5	$p \wedge q$ 6	$p \wedge r$ 7	$(p \wedge q) \vee (p \wedge r)$	$[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	T
T	F	T	T	T	F	T	T	T
T	F	F	F	F	F	F	F	T
F	T	T	T	F	F	F	F	T
F	T	F	T	F	F	F	F	T
F	F	T	T	F	F	F	F	T
F	F	F	F	F	F	F	F	T

15. $(p \vee q) \wedge \sim q \iff p \vee (q \vee \sim q)$ Law 7b

$\qquad\qquad\qquad \iff p \vee t$ Law 11a

$\qquad\qquad\qquad \iff t$ Law 12a

17. $p \wedge q \iff \sim[\sim(p \wedge q)]$ Law 1

$\qquad\quad \iff \sim(\sim p \vee \sim q)$ Law 9b

19. $(\sim p \wedge \sim q) \wedge \sim r \iff [\sim(p \vee q)] \wedge \sim r$ Law 9a

$\qquad\qquad\qquad\quad \iff \sim[(p \vee q) \vee r]$ Law 9a

$\qquad\qquad\qquad\quad \iff \sim[p \vee (q \vee r)]$ Law 7b

21. $p \wedge q \iff \sim[\sim(p \wedge q)]$ Law 1

$\qquad\quad \iff \sim[\sim p \vee \sim q]$ Law 9b

23. $p \to q \iff \sim p \vee q$ Law 3a

25. $p \vee q \iff \sim[\sim(p \vee q)]$ Law 1

$\qquad\quad \iff \sim[\sim p \wedge \sim q]$ Law 9a

27. $p \to q \iff \sim(p \wedge \sim q)$ Law 3b

29. Jim does not like Sue and does not like Mary.
31. Sales are not up, or we do not all get raises.
33. $\sim(p \to q) \iff \sim[\sim(p \wedge \sim q)]$ Law 3b

$\qquad\qquad\quad \iff p \wedge \sim q$ Law 1
35. I work hard, and I do not succeed.
37. Prices are raised, and sales do not drop.

L.5 EXERCISES

1.

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim p$	$[(p \vee q) \wedge \sim p] \to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

3. Valid

p	q	r	$p \to q$	$q \to r$	$(p \to q) \land (q \to r)$	$p \to r$	$[(p \to q) \land (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Valid

5.

p	q	$p \leftrightarrow q$	$(p \leftrightarrow q) \land q$	$[(p \leftrightarrow q) \land q] \to p$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

Valid

7.

p	q	r	$\sim q$	$p \to \sim q$	$r \to q$	$(p \to \sim q) \land (r \to q) \land r$	$\sim p$	$[(p \to \sim q) \land (r \to q) \land r] \to \sim p$
T	T	T	F	F	T	F	F	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	F	F	T
T	F	F	T	T	T	F	F	T
F	T	T	F	T	T	T	T	T
F	T	F	F	T	T	F	T	T
F	F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T	T

Valid

9.

p	q	r	$p \to q$	$p \to r$	$(p \to q) \land (p \to r)$	$q \land r$	$[(p \to q) \land (p \to r)] \to (q \land r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	F	T
T	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	F
F	F	T	T	T	T	F	F
F	F	F	T	T	T	F	F

Not valid

11.

s = "I go out with my sister" $\quad s \lor f \to \sim d$
f = "I go out with my friend" $\quad \underline{d}$
d = "I drive my car" $\quad\quad \therefore \sim s$

$$((s \lor f) \to \sim d) \land d \implies \sim(s \lor f) \quad \text{Modus tollens}$$
$$\iff \sim s \land \sim f \quad \text{De Morgan}$$
$$\implies \sim s \quad \text{Subtraction}$$

13. h = "You are happy" $\quad h \leftrightarrow \ell$
ℓ = "You are healthy" $\quad \ell \lor s$
s = "You are smart" $\quad \underline{\sim h}$
$\quad\quad \therefore \sim h \to \sim s$

In the case that h is false, ℓ false, and s true, $(h \leftrightarrow \ell) \land (\ell \lor s) \land (\sim h)$ is true while $\sim h \to \sim s$ is false. Thus the argument is not valid.

15.

e = "I have an egg for breakfast" $\quad\quad e \land j \to \sim r$
j = "I have orange juice for breakfast" $\quad \sim j \to \sim g$
r = "I have cereal for breakfast" $\quad\quad \underline{e \land g}$
g = "I have grapefruit juice for breakfast" $\quad \therefore \sim r$

$$(e \land h \to \sim r) \land (\sim j \to \sim g) \land (e \land g)$$
$$\iff (e \land j \to \sim r) \land (\sim j \to \sim g) \land g \land e \quad \text{Associativity}$$
$$\implies (e \land j \to \sim r) \land j \land e \quad \text{Modus tollens}$$
$$\iff (e \land j \to \sim r) \land (j \land e) \quad \text{Associativity}$$
$$\implies \sim r \quad \text{Modus ponens}$$

Valid.

17. r = "I clean my room" $\quad r \to \sim m$
m = "My mother is mad" $\quad \underline{m}$
$\quad\quad \therefore \sim r$

$$(r \to \sim m) \land m \implies \sim r \quad \text{Modus tollens}$$

Valid.

19. b = "Jane eats broccoli" $\quad b$
s = "Jane eats spinach" $\quad s$
$\quad\quad \underline{b \to \sim s}$
$\quad\quad \therefore s \land \sim b$

$$b \land s \land (b \to \sim s) \iff s \land b \land (b \to \sim s) \quad \text{Commutative law}$$
$$\implies s \land \sim s \quad \text{Modus ponens}$$
$$\iff c \quad \text{Inverse law}$$
$$\implies c \lor (s \land \sim b) \quad \text{Addition}$$
$$\iff s \land \sim b \quad \text{Identity law}$$

Valid.

21. b = "I play baseball" $\quad \sim b \to s$
s = "I play soccer" $\quad \underline{\sim s \to k}$
k = "I play basketball" $\quad \therefore \sim s \to b \lor k$

$$(\sim s \to b) \land (\sim s \to k) \iff (s \lor b) \land (s \lor k) \quad \text{Implication}$$
$$\iff s \lor (b \land k) \quad \text{Distribution}$$
$$\iff \sim s \to (b \land k) \quad \text{Implication}$$

Valid.

23.

$i =$ "The rate of inflation is increasing" $i \to g$

$g =$ "The price of gold is increasing" $g \to \sim b$

$b =$ "The price of bonds is increasing" $\dfrac{b}{\therefore \sim i}$

$$(i \to g) \wedge (g \to \sim b) \wedge b$$
$$\implies (i \to \sim b) \wedge b \quad \text{Hypothetical syllogism}$$
$$\implies \sim i \quad\quad\quad\quad\quad \text{Modus tollens}$$

Valid.

25. $s =$ "The price of sugar is rising" $s \to (d \wedge \ell)$

$d =$ "The price of candy is rising" $\sim d$

$\ell =$ "I eat less candy" $\dfrac{\sim \ell}{\therefore \sim s}$

$$(s \to (d \wedge \ell)) \wedge \sim d \wedge \sim \ell$$
$$\implies (s \to (d \wedge \ell)) \wedge (\sim(d \vee \ell)) \quad \text{De Morgan}$$
$$\implies \sim s \quad\quad\quad\quad\quad\quad\quad\quad \text{Modus tollens}$$

Valid.

27. $s =$ "I go out with my sister" $s \wedge f \to \sim d$

$f =$ "I go out with my friend" d

$d =$ "I drive my car" $\dfrac{}{\therefore \sim s}$

Assume $\sim s$ is false.
Then s is true.
Then $s \vee f$ is true.
If the first hypothesis is false, then we are done.
If the first hypothesis is true, then $\sim d$ is true.
Then the second hypothesis is false.

29. $i =$ "The rate of inflation is increasing" $i \to g$

$g =$ "The price of gold is increasing" $g \to \sim b$

$b =$ "The price of bonds is increasing" $\dfrac{b}{\therefore \sim i}$

Assume $\sim i$ is false.
Then i is true.
Suppose each hypothesis is true.
From hypothesis 1, g is then true.
From hypothesis 2, $\sim b$ is then true.
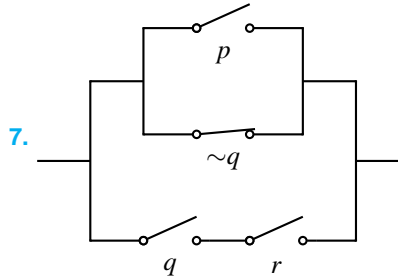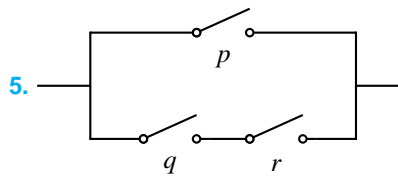This contradicts hypothesis 3.

L.6 EXERCISES

1. $p \vee p \iff p$. Current will flow when p is closed.

3.

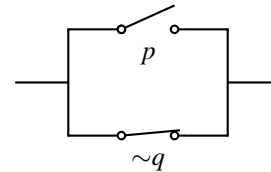| p	q	$\sim q$	$p \wedge \sim q$	$\sim p$	$\sim p \wedge q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	T	T	F	F	T
F	T	F	F	T	T	T
F	F	T	F	T	F	F

Current flows when one switch is open and the other is closed.

5.

7.

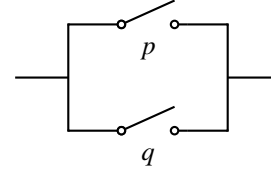**9.** $(p \wedge q) \vee p \iff p$ by absorption law.

11. $p \vee (\sim p \wedge \sim q) \vee (p \wedge q)$
$\iff (\sim p \wedge \sim q) \vee p$ Absorption law
$\iff (p \vee \sim p) \wedge (p \vee \sim q)$ Distribution law
$\iff t \wedge (p \vee \sim p)$ Inverse law
$\iff p \vee \sim q$ Identity law

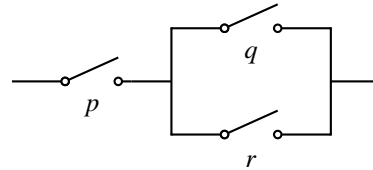Current flows if p is closed or q open.

13. $(p \vee q) \wedge (\sim q \vee q) \iff (p \vee q) \wedge t$ Inverse law
$\iff p \vee q$ Identity law

Current flows if either switch is closed.

15. $(p \wedge q) \vee (p \wedge r) \iff p \wedge (q \vee r)$ by the distributive law.
Current flows if p is closed and one of the others is closed.

REVIEW EXERCISES

1. Statement   **2.** Statement
3. Not a statement   **4.** Statement
5. a. Abraham Lincoln was not the shortest president of the United States.   **b.** Abraham Lincoln was the shortest president of the United States, and Carson City is the capital of Nevada.   **c.** Abraham Lincoln was the shortest president of the United States, or Carson City is the capital of Nevada.   **d.** Abraham Lincoln was not the shortest president of the United States, and Carson City is the capital of Nevada.   **e.** Abraham Lincoln was the shortest president of the United States,

or Carson City is not the capital of Nevada.   **f.** It is not true that Abraham Lincoln was the shortest president of the United States and Carson City is the capital of Nevada.

6. a. Abraham Lincoln was not the shortest President of the United States. Carson City is not the capital of Nevada.   **b.** Abraham Lincoln was the shortest president of the United States, or Carson City is the capital of Nevada.   **c.** Abraham Lincoln was the shortest president of the United States, and Carson City is the capital of Nevada.

7. a. $\sim q$ **b.** $p \vee q$ **c.** $p \vee \sim q$ **d.** $\sim p \wedge \sim q$

8.

p	q	$\sim p$	$\sim p \wedge q$	$\sim p \vee q$	$(\sim p \wedge q) \wedge (\sim p \vee q)$
T	T	F	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	F	T	F

9.

p	q	r	$\sim r$	$p \vee q$	$q \wedge \sim r$	$(p \vee q) \vee (q \wedge \sim r)$
T	T	T	F	T	F	T
T	T	F	T	T	T	T
T	F	T	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	T	F	T	T	T	T
F	F	T	F	F	F	F
F	F	F	T	F	F	F

10.

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim p \vee q$	$(p \vee \sim q) \vee q \vee (\sim p \vee q) \vee \sim q$
T	T	F	F	T	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

11.

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \wedge q$	$\sim p \wedge q$	$[(p \wedge \sim q) \wedge q] \vee (\sim p \wedge q)$
T	T	F	F	F	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	F	F	T	T
F	F	T	T	F	F	F	F

12. a. True   **b.** True   **c.** False   **d.** True

13. If inflation is increasing, then the price of gold is rising. *Contrapositive*: $\sim q \rightarrow \sim p$. If the price of gold is not rising, then inflation is not increasing. *Inverse*: $\sim p \rightarrow \sim q$. If inflation is not increasing, then the price of gold is not rising. *Converse*: $q \rightarrow p$. If the price of gold is rising, then inflation is increasing. **b.** If inflation is increasing, then the price of gold is not rising. *Contrapositive*: $q \rightarrow \sim p$. If the price of gold is rising, then inflation is not increasing. *Inverse*: $\sim p \rightarrow q$. If inflation is not increasing, then the price of gold is rising. *Converse*: $\sim q \rightarrow p$. If the price of gold is not rising, then inflation is increasing. **c.** If the price of gold is not rising, then inflation is not increasing. *Contrapositive*: $p \rightarrow q$. If inflation

is increasing, then the price of gold is rising. *Inverse*: $q \rightarrow p$. If the price of gold is rising, then inflation is increasing. *Converse*: $\sim p \rightarrow \sim q$. If inflation is not increasing, then the price of gold is not rising.

14.

p	q	$\sim p$	$\sim p \rightarrow q$	$\sim(\sim p \rightarrow q)$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	F
F	F	T	F	T

15.

p	q	r	$p \wedge q$	$p \wedge q \rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

16.

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

17.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \wedge q$	$(p \wedge q) \rightarrow r$	$[p \rightarrow (q \rightarrow r)] \rightarrow [(p \wedge q) \rightarrow r]$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

18.

p	q	$p \wedge q$	$p \vee (p \wedge q)$	$[p \vee (p \wedge q)] \leftrightarrow p$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	F	T

19.

p	q	$p \rightarrow q$	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$	$(p \rightarrow q) \leftrightarrow [\sim(p \wedge \sim q)]$
T	T	T	F	F	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

20. a. Sue will not major in math and will not major in

physics.   **b.** Sue does not enjoy math or does not enjoy physics.

21.

p	q	r	$\sim q$	$\sim r$	$p \to \sim q$	$\sim r \to p$	$q \wedge (p \to \sim q)$ $\wedge (\sim r \to p)$	$q \wedge (p \to \sim q)$ $\wedge (\sim r \to p) \to r$
T	T	T	F	F	F	T	F	T
T	T	F	F	T	F	T	F	T
T	F	T	T	F	T	T	F	T
T	F	F	T	T	T	T	F	T
F	T	T	F	F	T	T	T	T
F	T	F	F	T	T	F	F	T
F	F	T	T	F	T	T	F	T
F	F	F	T	T	T	F	F	T

Valid

22. $s =$ "It is sunny" $s \to \sim d$

$d =$ "I study" $\sim s \to \sim b$

$b =$ "I am at the beach" $\underline{s \wedge b}$

$\hspace{6cm} \therefore \sim d$

$(s \to \sim d) \wedge (\sim s \to \sim b) \wedge (s \wedge b)$

$\quad \Longrightarrow \ (s \to \sim d) \wedge (\sim s \to \sim b) \wedge s \wedge b$ Associative

$\quad \Longrightarrow \ (s \to \sim d) \wedge s$ Subtraction

$\quad \Longrightarrow \ s \wedge (s \to \sim d)$ Commutative

$\quad \Longrightarrow \ \sim d$ Modus ponens

Valid.

23. $[(p \wedge \sim q) \vee q \vee (\sim p \wedge q)] \wedge \sim q \iff \sim p \vee \sim q$

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