Math 664-600, TR 2:20 - 3:35, Blocker 160
Periodic ordinary and partial differential equations and their applications

Instructor
Peter Kuchment, Blocker 614A, kuchment@math.tamu.edu

1. Course Description:

ODEs and PDEs with periodic coefficients have been arising in mathematics and its various applications (mechanics, fluid mechanics, solid state physics, material science, nano-materials, optics, electrical engineering, etc.) more than a century. The main components of the theory of such ODEs were developed at the end of 19th [14, 23, 24] and in the first half of 20th century, while another significant outburst of the progress happened in 1970s, after the explosion of the exactly integrable systems theory.

The even more important (and much harder) PDE counterpart also has a long history, starting with the solid state theory in 1930s. Due to many new applications, it has been being developed through the last decades, and many parts of the theory are not finished yet.

For PDEs, the most comprehensive current sources available are the instructor's survey in the Bulletin of the AMS in July 2016 and a much more expanded book by him in preparation. Se.e also some references below.

Besides appearing in many applications, periodic media share some important common features concerning waves propagation, which justify studying them together.

The course will introduce the main notions, results, and approaches both for the ODE and PDE situations. In particular, relations to stability of periodic motions, spectral theory, wave propagation, and solid state physics will be described.

2. The recommended prerequisites

Basic knowledge of ODEs and PDEs, Fourier transform, as well as of real, complex, and functional analysis (or the instructor's consent).

3. Textbook

No textbook is required, attendance and distributed lecture notes will be sufficient.

4. Tentative plan

All topics below will be provided with physics and engineering examples and motivations.
The main emphasis will be on the items (2) through (4) below:

(1) Motivation
(2) 1st order periodic linear systems. Main notions. Floquet/Lyapunov theorems.
(3) Hill's equation. Spectral theory. Spectral bands and gaps (stop bands). Dispersion relation, etc.
(4) Periodic linear elliptic PDEs. Spectral theory.
(5) Time-periodic parabolic PDEs.
(6) Time-periodic hyperbolic PDEs.

5. Grading

will be based upon attendance, class participation, and assigned class presentations. **Non-mandatory** home problems might be suggested . . . *periodically* .

6. Recommended supplementary literature

The literature on the subject is vast and hard to list. Here are some of the useful sources:

- [30] is an outstanding (and insufficiently well known) treatise on periodic ODEs and systems of ODEs and their various applications.
- The beautiful old book [5] (and its extended French edition [6]), written by a Nobel prize winner, is still a very good and readable source for understanding the role of periodicity, in many its incarnations, in wave propagation.
- [1,2,8] are great classical textbooks on ODEs with chapters on periodic ODEs/Floquet theory.
- [3, 7, 13, 25] are books completely devoted to periodic ODEs (some also involve PDEs). See also [10].
- Chapters devoted to various topics on periodic ODEs and PDEs can be found in [26, 27, 29]
- Periodic PDEs are considered in detail in [19], but this book, written (badly) by the instructor, is **not recommended** for first reading (at least the 1st chapter should be skipped). The survey [21] is better. See also [17, 28] and the last part in [27].
- Relations to photonic crystals are explained in [12, 16, 20].
- About relations to fluid dynamics, see, e.g. [9, 31].

- Discussions of the graph versions and their applications to the nano-materials can be found in [4, 11, 18, 22].
- Electrons in strong monochromatic external fields are considered, e.g. in [15, 32]
- Relations to the inverse scattering method are discussed in [26].

References

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