Leon Ehrenpreis, recollections from the recent decades

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Leon Ehrenpreis was an outstanding world class mathematician and a wonderful, warm person. I had a privilege to consider myself his friend for the last two decades. It is hard to do justice to his manifold mathematics and personality, but I will try to at least add some recollections to this tribute volume[1].

Leon Ehrenpreis has been one of my mathematical heroes for about 40 years. I first encountered his, Lars Hörmander's, Bernard Malgrange's and Victor Palamodov's fundamental and beautiful works on systems of linear constant coefficient PDEs in early 1970s, when I was an undergraduate student and then a PhD candidate. They have had a profound impact on me, in particular when working on the Floquet theory of periodic PDEs, which we with Leonid Zelenko started developing in a few years. I am sure that Bernard Malgrange and Daniele Struppa have described this part of Leon's legacy much better than I ever could. I will only address some of the research Leon pursued in the last two decades of his life, which I was lucky to witness.

Some time around 1988, a medical industry contract forced me to learn the basics of a fascinating topic that I had never heard of before, the so called computed tomography. This turned out to be fateful. Our research group in Voronezh found the mathematics of tomography so challenging and exciting, that in the following decades several of us have being devoting a significant part of time working on tomographic problems. Appearance in the 80s of the Russian translation of the cornerstone book on this topic by Frank Natterer [52] also helped. Interestingly enough, I discovered that several mathematicians whom I admired for their work in completely different

[1] One can also read the AMS Notices article [30] for recollections of several Leon's friends and colleagues. A volume on tomography [10] to appear soon is also dedicated to Leon's memory.

areas (e.g., Carlos Berenstein, Simon Gindikin, and Victor Palamodov) had already been working on tomography related issues. This is an instance of a strange effect that I have observed several times in my life, when several people working in closely related areas suddenly and independently make a sharp turn to the same new direction.

The end of 1980s was a fascinating time in the former Soviet Union, when contacts with the West have started to become somewhat possible. In particular, the existence of some of famous mathematicians could be checked experimentally (before that, the names like L. Ehrenpreis, L. Hörmander, P. Lax, L. Nierenberg, and many others seemed to me to belong to some deities rather than real people). In 1989 I had my first chance to travel abroad, and I spent about a month in the USA going to various universities and to an AMS tomography conference in Arcata, California. This is where I saw for the first time some of my scientific heroes (e.g., L. Ehrenpreis, S. Helgason, F. Natterer) in flesh[2]. Meeting Leon in Arcata was a big surprise to me, since I had no clue that he had become interested in integral geometry or tomography. This was another instance of the simultaneous change of direction. He showed a polite interest in what I told him about my PDE work related to his, but it was clear that he was thinking in somewhat different (although not orthogonal) direction now. This was the first time when I heard Leon mentioning his book on Radon transform, which was "nearly finished." It did appear ... in 2003 [28]. In the 13-14 years in between, Leon had been sending generously the nth versions of his manuscript to anyone interested, and the ideas and problems contained in these texts have influenced many of us.

After emigrating later in 1989 to the USA, I found employment at the Wichita State University in Kansas. The year 1990 was a tough time for finding employment for a middle-age emigree mathematician with mediocre, at best, command of English. Having recommendation letters from colleagues such as L. Ehrenpreis was crucial, and I am indebted forever to them and many other mathematicians who supported me in various ways in these difficult times.

[2]The Arcata meeting was also the place where I met for the first time other colleagues, whom I now consider as long time friends (J. Boman, D. Finch, A. Markoe, E. T. Quinto, G. Uhlmann, and many others). I could not even imagine that twelve years later I would have a privilege to work at the same department at Texas A&M with another group of researchers whose work I studied and admired as a young mathematician in Russia, such as Ron Douglas, Ciprian Foias, Carl Pearcy, and Gilles Pisier.

Settling down in Wichita was rather pleasant. My family loved the city. The mathematics department was quite good, including several prominent people in the areas of my interest, in particular in inverse problems (Victor Isakov and Ziqi Sun). When I started bringing in speakers and collaborators, Leon Ehrenpreis was one of the first invitees, and since then he had become a constant visitor of our department and then of the Mathematics department of Texas A&M, where I moved in 2001. His lectures and discussions that I and my graduate students had with him were extremely interesting, scientifically rewarding, and personally enjoyable.

I will skip some personal recollections, which one can find in [30] and concentrate rather on mathematics. One of the first topics that we discussed was a strange byproduct of the papers [48, 49] published a couple of years before. There we with S. Lvin described the range of the so called exponential Radon transform, which arises in the Single Photon Emission Computed Tomography (SPECT), an important medical imaging method [52]. I will not burden the reader with technicalities and just describe the result on a hand-waving level. It is known [34–36, 42, 43, 52] that the ranges of Radon type transforms are usually of infinite co-dimension in natural function spaces. Knowing the description of the range plays an important role in integral geometry and tomography. After range conditions are found, it is usually straightforward to go back and check their necessity[3], while proof of their completeness is usually technical. Thus, when the conditions of [48] were found, we expected that re-proving their necessity should a piece of cake: just plug the transform of a function into these conditions and see immediately that they are satisfied. However, when we did this, we discovered an infinite and totally non-obvious to us set of non-linear differential identities for the standard sine function: for any odd natural n,

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{dx} - \sin x\right)\circ\left(\frac{d}{dx} - \sin x + i\right)\circ\ldots\circ\left(\frac{d}{dx} - \sin x + (k-1)i\right)\left((\sin x)^{n-k}\right) = 0, \tag{1}$$

where i is the imaginary unit and $\circ$ denotes the composition of differential operators. The attempt to prove these identities directly (i.e., without any integral geometry and Fourier analysis) succeeded [49], but took a significant

[3]For instance, when the so called **moment conditions** [35, 42] for the standard Radon transform are written, checking their necessity boils down to noticing that the kth power $(x \cdot \omega)^k$ of the inner product of two vectors is a homogeneous polynomial of degree k with respect to each of them.

time. We are still puzzled by the meaning of these identities [50]. Several integral geometry and tomography experts devoted their time and effort to trying to understand better the meaning of these range conditions. This is also what we set out to do with my PhD student Valentina Aguilar and Leon Ehrenpreis. We succeeded in the following sense: we showed, in particular, that these identities are equivalent to an interesting theorem of separate analyticity type.

Theorem [8]
Let D be a disk in $\mathbb{R}^2$ and f be a function in the exterior of D. Suppose that when restricted to any tangent line L to D, the function $f|_L$, as a function of one real variable extends to an entire function on the complexification of L. Then f, as a function on $\mathbb{R}^2 \setminus D$ extends to an entire function on $\mathbb{C}^2$.

Well, this fact also did not look obvious to us. Analyticity of f in a complex neighborhood of $\mathbb{R}^2 \setminus D$ follows from the old (and not that well known) separate analyticity theorem by S. Bernstein (see [9]), however this theorem cannot produce statement about f being an entire function. Thus, since proving the above theorem, a couple of things about it kept bothering us for several years. First of all, this is a several complex variables fact, while our proof did not look like a SCV argument at all. Is there a truly complex analysis proof? Another, related, question is whether such a theorem can be proven for a different convex body instead of a disk D? A SCV proof was later provided in [54], although it was rather complicated and was not generalizable (at least, easily) to other convex curves. Leon has worked out some other examples of convex algebraic curves (unpublished), but general picture remained unclear to us. Finally, A. Tumanov presented recently [59] a beautiful short proof based on attachment of analytic disks (where Tumanov is a great expert), which works for any strictly convex body D with a mild conditions on the smoothness of its boundary.

Another issue that we addressed with Leon and my Masters student Alex Panchenko, also originated from emission tomography. The exponential Radon transform in SPECT depends upon an "attenuation" parameter $\mu \geq 0$. In [29], we introduced and studied a "mother" exponential Radon transform, which had no free parameters, but by different restrictions of which one can obtain the exponential Radon transforms corresponding to all possible values of the attenuation. We also obtained the range description there, which was based upon the F. John's differential equations. In this particular case, the (ultra-hyperbolic) John's equation could be recast as a boundary Cauchy-Riemann equation.

Although we have not done any joint research since 2000, we kept discussing (in person and by e-mail) various integral-geometric and PDE issues. One was the fascinating and surprisingly hard "strip problem" [1, 2, 4, 6, 37, 38, 57, 58], to which Leon has contributed [27, 28] and which it extended to a more general PDE setting (see, e.g., [6,27,28]). It was eventually resolved due to efforts of several mathematicians, including M. Agranovsky, J. Globevnik, and A. Tumanov (see the reference above).

Leon was also very much interested in the activity concerning the "restricted spherical means" operator, i.e. a version of Radon transform that integrates a given function over spheres of arbitrary radii, but with the centers restricted to a hyper-surface S. The study of such operators was very active since the beginning of 1990s, due first to needs of approximation theory , then self-sustained just due to the beauty and complexity of arising problems (see [3, 7] and references therein), and finally it received a huge boost in the last decade, due to the discovered relations to a newly developing method of medical imaging, the so called thermo-/photo-acoustic tomography (see the surveys [5, 31–33, 47, 60] and references there).

The restricted spherical mean problem happens to be a very particular case of one of the questions raised by Leon in his book [28]. This brings us from the "small" problems discussed above to the much more general thinking Leon has been doing on transforms of Radon type and their very wide generalizations. This was reflected in his papers of the period and in the monograph [28]. The title of this book,"The Universality of the Radon Transform," and the wealth of topics and ideas covered and variety of open problems suggested shows how deeply Leon believed in wide range importance of this approach. He was not the first to realize such widespread applicability of transforms of Radon type, although probably the first to give such a bold name to a book. Fritz John in his book [46] showed how important this circle of ideas is for PDEs. Israel Gelfand, Simon Gindikin, Sigurdur Helgason, Victor Palamodov, and many other mathematicians studied in detail applications to PDEs, harmonic analysis, group representation theory, special functions, mathematical physics, etc. (e.g., [34–36, 42–45, 55]). Still, Leon's book is rather unique in terms of many non-standard issues raised there. Leon also was unique in his writing style, introducing new notations and names for well known objects, which does not help a reader. However, after getting through these hurdles, one opens a treasure chest of ideas.

The variety of things that Leon addressed in the book [28] and his other publications of the time ([11]- [27]), and which he considered inter-related, is enormous: "exotic" boundary value problems for PDEs, Poisson summation formulas, Eisenstein and Poincare series on $SL(2,\mathbb{R})$ and $SL(3,\mathbb{R})$, various number theoretic problems, Hartogs-Lewy extension, FBI transform (although it carries an unrecognizable name in [28], being an instance of what he called "nonlinear Fourier transform"), edge-of-the-wedge theorems, Phragmén-Lindelöf type theorems for PDEs, special functions, among others.

Notwithstanding the overarching title, a wide variety of topics covered, and large volume, [28] is neither a textbook on the "usual" Radon transform, nor a comprehensive historical survey or reference manual; it is not designed for reading by an uninitiated; it does not cover many important developments, techniques, and results that one can find in [34–36, 39–45, 55, 56], such as curved manifolds case, κ-operator approach, Radon transforms of differential forms and tensors, projective geometry setting, most of the group representation relations, etc. At Leon's request, Todd Quinto and I contributed the appendix [51] to [28] devoted to a brief survey of some tomographic applications. Due to the natural size limitations, it also cannot be considered comprehensive. One can find a thorough discussion of tomographic issues in [52, 53].

In spite of all these omissions, this unique book [28] should occupy a space on the bookshelf of anyone working on PDEs, Fourier analysis, several complex variables, and integral geometry. I am sure it will be a source of inspiration for many mathematicians, who will take their time to get through the text.

The memory of Leon Ehrenpreis will stay with all who encountered his amazing mathematics and experienced his friendship. I am grateful to the fate for giving me the chance and privilege to meet Leon and to collaborate with him.

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