Amer. Math. Soc. Transl.
Volume **231**, 2010

# On Guided Electromagnetic Waves in Photonic Crystal Waveguides

Peter Kuchment and Beng-Seong Ong

*Dedicated to the memory of Professor V. B. Lidskii*

Abstract. This paper addresses the issue of existence and confinement of electromagnetic modes guided by linear defects in photonic crystals. Sufficient conditions are provided for the existence of such waves near a given spectral location. Confinement to the guide is achieved due to a photonic band gap in the bulk dielectric medium.

## 1. Introduction

A photonic crystal, also called photonic band gap (PBG) material, is a periodic medium which plays the role of an optical analog of a semiconductor. Such a medium has a gap in the frequency spectrum of electromagnetic (EM) waves. The idea of a photonic crystal was first suggested in 1987 [**17**, **31**] and has since been intensively studied experimentally and theoretically (see, e.g., the recent books [**16**,**18**,**26**,**27**,**32**], the mathematical survey [**21**], the online bibliography [**24**], and references therein). This interest has been triggered by the numerous promising applications of PBG materials, one of which is using photonic crystal for manufacturing highly efficient optical waveguides. The idea is to introduce a linear "defect" into a PBG material and to guide through it EM waves of a frequency prohibited in the bulk. Numerical and experimental studies have shown that such superior guides can be efficiently created, e.g., [**16**, **18**, **23**, **26**, **27**].

In order to create such a guide, one needs to establish several facts. The first, and foremost, is existence of guided waves of frequencies in the band gap. The second, and an easier one, is confinement of these modes. This paper addresses both issues, by finding some sufficient conditions of existence of guided modes and showing their confinement to the guide, in the sense of being evanescent in the bulk.

2010 *Mathematics Subject Classification.* Primary 35P99, 35Q60; Secondary 78A48.

*Key words and phrases.* Photonic crystal, defect, waveguide, spectrum, Maxwell operator, guided mode.

Work of both authors was partially supported by the NSF Grants DMS 0296150 and 9971674. Work of the first author was also supported by the NSF Grant DMS 0072248 and was also partially supported by the KAUST grant KUS-CI-016-04 through the IAMCS. The authors thank the NSF, IAMCS, and Kaust for the support.

Similar results were previously obtained by the authors in [**22**] for scalar models (i.e., for acoustic analogs of PBG waveguides). In this paper, we will address the above questions for the full Maxwell case.

There is another important question to be resolved. Namely, one needs to show that the impurity spectrum that arises in the spectral gaps due to the presence of a linear defect does not correspond to bound states. This difficult issue is not addressed here (see some relevant remarks and references in Section 5).

In Section 2 we introduce the main model to be investigated. Section 3 contains formulation of the main results, with the proofs provided in Section 4. The paper ends with the sections devoted to final remarks and acknowledgments, as well as the bibliography.

## 2. Preliminaries

We start by describing the mathematical model studied in this paper. Let $\varepsilon_0(x)$ be a bounded positive measurable functions in $\mathbb{R}^3$ separated from zero:

$$0 < c_0 \le \varepsilon_0(x) \le c_1 < \infty. \tag{2.1}$$

It is usually assumed in photonic crystal theory that $\varepsilon_0$ is periodic with respect to a lattice $\Gamma \subset \mathbb{R}^3$, but this is not required for our results.

The function $\varepsilon_0$ represents the dielectric properties of the bulk material. In other words, one can think of the space $\mathbb{R}^3$ filled with a dielectric material with the dielectric function $\varepsilon_0$.

We will use the name "unperturbed Maxwell operator" for the selfadjoint realization of the operator

$$M_0 := \nabla^\times \frac{1}{\varepsilon_0(x)} \nabla^\times \tag{2.2}$$

in $L_2(\mathbb{R}^3; \mathbb{C}^3)$ defined by means of its quadratic form

$$\int \varepsilon_0^{-1} |\nabla^\times u|^2 dx \tag{2.3}$$

with the domain $H^1(\mathbb{R}^3; \mathbb{C}^3)$. We use here the shorthand notation

$$\nabla^\times u = \nabla \times u = \text{curl } u.$$

Usually the name "Maxwell equations" is reserved for the well-know first-order system [**15**]. However, in the monochromatic case it reduces to the spectral problem for the operator (2.2).

A cylindrical domain $\mathcal{S}_l$ (see Figure 1) will represent a linear "defect strip":

$$\mathcal{S}_l := \{x = (x_1, x') \in \mathbb{R}^3 \,|\, x_1 \in \mathbb{R},\, x' \in l\Omega\}.$$

Here the cross-section $l\Omega$ of the strip is a domain $\Omega$ in $\mathbb{R}^2$ (e.g., the unit ball centered at the origin), scaled with factor $l$.

We now introduce the perturbed medium with homogeneous dielectric properties inside the defect strip $\mathcal{S}_l$:

$$\varepsilon(x) = \begin{cases} \varepsilon > 0 & \text{for } x \in \mathcal{S}_l, \\ \varepsilon_0(x) & \text{for } x \notin \mathcal{S}_l, \end{cases} \tag{2.4}$$

where $\varepsilon$ is a constant.

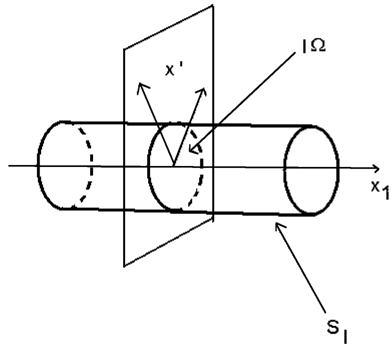


FIGURE 1. The waveguide $\mathcal{S}_l$.

The perturbed Maxwell operator

$$M := \nabla^\times \frac{1}{\varepsilon(x)} \nabla^\times \tag{2.5}$$

corresponds to the medium with the linear defect. It is defined, analogously to $M_0$, as a selfadjoint operator in $L_2(\mathbb{R}^3; \mathbb{C}^3)$.

REMARK 2.1. The reader has probably noticed that we disregard the standard restriction of the Maxwell operator to divergence-free fields [**16**]. The difference is essentially in acquiring a huge eigenspace corresponding to the zero frequency, with all other parts of the spectral decomposition staying intact. Since the problem of guided waves concerns the situation inside the spectral gaps of the operator $M_0$, this difference is irrelevant in this case. On the other hand, abandoning the zero divergence condition will simplify the techniques considerably.

Our goal is the same as in the paper [**22**] devoted to the scalar (acoustic) case, i.e., to show that for any gap $(\alpha, \beta)$ in the spectrum $\sigma(M_0)$ of the unperturbed medium, under appropriate conditions on the parameters $l$ and $\varepsilon$ of the line defect, additional spectrum arises inside the gap, with the corresponding (generalized) eigenmodes being confined to the defect (evanescent in the bulk).

## 3. Formulation of the results

In order to formulate our first main result, we need to introduce the following quantity:

DEFINITION 3.1. We denote by $\nu > 0$ the lowest eigenvalue of the Laplace operator $\Delta$ acting on divergence-free $\mathbb{R}^2$-valued vector fields on $\Omega$ with Dirichlet boundary conditions on $\partial\Omega$.

I.e., $\nu > 0$ is the smallest number for which a nontrivial solution of the following problem exists:

$$\begin{cases} -\Delta E = \nu E \text{ in } \Omega, \\ \nabla \cdot E = 0 \text{ in } \Omega, \\ E\,|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where $\frac{\partial}{\partial n}$ is the external normal derivative on $\partial\Omega$.

Our main results are given in the following theorems.

THEOREM 3.2. *Let $G = (\alpha, \beta)$ be a nonempty finite interval, such that $\alpha > 0$ (we will be especially interested in the case when $G$ is a gap in the spectrum of the "background medium" operator $M_0$). Let the following inequality be satisfied:*

$$l^2(\beta - \alpha)\varepsilon > 2\nu. \tag{3.2}$$

*Then the interval $G$ contains at least one point of the spectrum $\sigma(M)$ of the perturbed operator.*

This theorem guarantees that when (3.2) is satisfied, eigenmodes of the perturbed medium do arise in the spectral gaps of the background medium. Furthermore, if $\delta > 0$ is such that $l^2\delta\varepsilon > \nu$, the corresponding spectrum forms a $\delta$-net in the gap.

Before one can fully associate these modes with the guided waves, one needs to establish their confinement to the waveguide (i.e., their evanescent nature in the bulk of the material). In order to describe the corresponding result, we need to remind the reader about some notions and results concerning generalized eigenfunction expansions. Here by generalized eigenfunctions one understands solutions of the eigenvalue problem, which do not decay sufficiently fast (or do not decay at all) to belong to the ambient Hilbert space $L^2$. One can find detailed discussions of generalized eigenfunction expansions, for instance, in [**3**, **4**].

Namely, as is shown in [**19**], for the operator $M$ that we consider, for almost any (with respect to the spectral measure) $\lambda$ in the spectrum $\sigma(M)$, there is a generalized eigenfunction $u_\lambda(x) \in H^1_{loc}(\mathbb{R}^3, \mathbb{C}^3)$ with the growth estimates

$$(1 + |x|)^{-N} u(x) \in L_2(\mathbb{R}^3; \mathbb{C}^3) \text{ and } (1 + |x|)^{-N}\, \nabla^\times u(x) \in L_2(\mathbb{R}^3; \mathbb{C}^3) \tag{3.3}$$

for some $N > 0$ (that might depend on the eigenfunction). This system of generalized functions is complete in the whole space (see [**3**, **4**, **19**] for detailed explanations of the meaning of this completeness). For elliptic operators with smooth coefficients, this is a well-known fact [**3**].

DEFINITION 3.3. We refer to the following growth condition as *polynomial boundedness of order $N$*: for any compact set $K \subset \mathbb{R}^3$ and $x \in \mathbb{R}^3$,

$$\|u\|_{L_2((K+x);\mathbb{C}^3)} + \|\nabla^\times u\|_{L_2((K+x);\mathbb{C}^3)} \le C_K(1 + |x|)^N. \tag{3.4}$$

In the next result we will use the following notation: for $x = (x_1, x') \in \mathbb{R}^3$, we denote by $\chi_x(y)$ the *characteristic function* of the cube

$$\{y \mid |y_j - x_j| \le 1 \text{ for } j = 1, 2, 3\}$$

centered at $x$, i.e., $\chi_x(y)$ is equal to 1 when $y$ is in this cube and 0 otherwise.

THEOREM 3.4. *Let $G$ be a finite spectral gap of $M_0$ and let $u_\lambda$ be a polynomially bounded generalized eigenfunction of $M$ corresponding to $\lambda \in G \cap \sigma(M)$. Then there exist positive constants $C$ and $C(\lambda)$ such that*

$$\|\chi_x u_\lambda\| \le C\,(1 + |x_1|)^N\, e^{-C(\lambda)\mathrm{dist}(x,\, S_l)}, \tag{3.5}$$

*where $N$ is the order of polynomial boundedness of $u_\lambda$.*

When the bulk medium is periodic in the $x_1$-direction, the polynomial growth in (3.5) disappears:

THEOREM 3.5. *If $\varepsilon_0(x)$ is periodic in the $x_1$-direction, then one can find a complete family of generalized eigenfunctions that satisfies*

$$\|u_\lambda\|_{L_2((K+x);\mathbb{C}^3)} + \|\nabla^\times u_\lambda\|_{L_2((K+x);\mathbb{C}^3)} \le C_K(1+|x'|)^N \tag{3.6}$$

*for any compact set $K \subset \mathbb{R}^3$, $x \in \mathbb{R}^3$. In this case, for $\lambda \in G \cap \sigma(M)$, one has the estimate*

$$\|\chi_x u_\lambda\| \le Ce^{-C(\lambda)\operatorname{dist}(x,\mathcal{S}_l)}. \tag{3.7}$$

## 4. Proofs of the results

In what follows, the norm and inner product in $L_2(\mathbb{R}^n;\mathbb{C}^3)$ will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

**4.1. Proof of Theorem 3.2.** We will show that if $\mu > 0$ and $\delta > 0$ is such that $l^2\delta\epsilon > \nu$, there is some spectrum of the operator $M$ in the $\delta$-vicinity of $\mu$. Then, taking $\mu = (\alpha+\beta)/2$ and $\delta = (\beta-\alpha)/2$, one gets the statement of the theorem.

In order to show this, due to selfadjointness of $M$, it is sufficient to find a vector function $w \in L_2(\mathbb{R}^3,\mathbb{C}^3)$ of unit norm, such that $w \in \mathcal{D}(M)$ and

$$\|Mw - \mu w\|^2 < \delta^2. \tag{4.1}$$

Let $g$ be a smooth, unit $L^2$-norm, divergence-free real vector field on $\mathbb{R}^2$ with compact support in $\Omega$ and unit $L^2$-norm, i.e, $g(y,z) = (\phi(y,z),\zeta(y,z))$ where $\phi$, $\zeta \in C_0^\infty(\Omega)$ and $\nabla\cdot g = 0$. We define

$$g_l := (\phi_l,\zeta_l) = l^{-1}(\phi(x'/l),\zeta(x'/l)).$$

Then $g_l$ is also a unit $L^2$-norm divergence-free field. Let $\psi(x_1) \in C_0^\infty(\mathbb{R})$ have unit $L_2(\mathbb{R})$-norm and let $\psi_n(x_1) = n^{-1/2}\psi(x_1/n)$ for $n > 0$. Clearly $\psi_n(x_1)$ also has unit $L_2$-norm.

Denoting $k = \sqrt{\mu\varepsilon}$, we introduce the following candidate for an approximate eigenfunction:

$$w_{l,n}(x) = \psi_n(x_1)e^{ikx_1}\begin{pmatrix} 0 \\ \phi_l(x') \\ \zeta_l(x') \end{pmatrix}. \tag{4.2}$$

The function $w_{l,n}$ clearly has unit norm in $L_2(\mathbb{R}\times l\Omega)$.

Instead of estimating the left-hand side of inequality (4.1), we will estimate $\|\varepsilon\,(Mw-\mu w)\|^2$. Taking into account that the function $w$ is supported inside the defect, the needed inequality (4.1) can also be rewritten as

$$\|\nabla^\times\nabla^\times w - k^2 w\|^2 < \delta^2\varepsilon^2. \tag{4.3}$$

Using the identity

$$\nabla^\times\nabla^\times w = -\Delta w + \nabla(\nabla\cdot w)$$

and the fact that $g_l$ is divergence-free, we obtain

$$\|\nabla^\times\nabla^\times w - k^2 w\|^2 = \left\| \begin{pmatrix} 0 \\ -(\psi_n'' + 2ik\psi_n')\phi_l - \psi_n\Delta\phi_l \\ -(\psi_n'' + 2ik\psi_n')\zeta_l - \psi_n\Delta\zeta_l \end{pmatrix} \right\|^2$$

where the norms are in $L_2(\mathbb{R}^3;\mathbb{C}^3)$.

Since the functions $\phi$, $\zeta$, and $\psi$ are real valued, their assumed normalization shows that the above expression is equal to

$$n^{-4}\,\|\psi''\|^2_{L_2(\mathbb{R})} + 4k^2n^{-2}\,\|\psi'\|^2_{L_2(\mathbb{R})} + l^{-4}\,\|\Delta g\|^2_{L_2(\Omega,\mathbb{R}^2)} + 2(nl)^{-2}\langle\psi'',\psi\rangle_{L_2(\mathbb{R})}\langle\Delta g, g\rangle_{L_2(\Omega,\mathbb{R}^2)}.$$

Since $n$ can be chosen arbitrarily large, the terms with the factors that are negative powers of $n$ can be made arbitrarily small (uniformly with respect to $k$ on any finite interval). Hence, one needs to control only the remaining terms by an appropriate choice of a divergence-free vector field $g$. In other words, one is interested in making $l^{-4}\,\|\Delta g\|^2_{L_2(\Omega,\mathbb{R}^2)}$ smaller than $\delta^2\varepsilon^2$, i.e.,

$$\|\Delta g\|_{L_2(\Omega;\mathbb{R}^2)} < l^2\delta\varepsilon, \tag{4.4}$$

while keeping $\langle\Delta g, g\rangle$ under control.

Since

$$\nu = \inf\|\Delta g\|_{L_2(\Omega;\mathbb{R}^2)},$$

where the *infimum* is taken over real, unit $L^2$-norm, divergence-free vector fields $g \in C_0^\infty(\Omega;\mathbb{R}^2)$, this condition boils down to

$$l^2\delta\varepsilon > \nu, \tag{4.5}$$

which proves the statement of the theorem. □

**4.2. Proof of Theorem 3.4.** We assume here that $\lambda$ belongs to a finite gap $G$ of the spectrum of $M_0$ and that $u := u_\lambda$ is the corresponding generalized eigenfunction of $M$ satisfying (3.4). Let us also introduce the resolvent $R(\lambda) = (M_0 - \lambda)^{-1}$. We will also use the function $\chi_x$ introduced before Theorem 3.4.

We will need the following auxiliary statement concerning the exponential decay of the resolvent, which is a result of [**8**]:

LEMMA 4.1 ( [**8**]). *There exists a positive number $m_\lambda$ that depends only on the distance of the point $\lambda$ from the gap edges, such that for a positive constant $C$, the following estimates hold for the local $L_2(\mathbb{R}^3;\mathbb{C}^3)$-norm of the resolvent $R(\lambda)$:*

$$\begin{aligned} \|\chi_u R(\lambda)\chi_v\| &\le Ce^{-m_\lambda|u-v|}, \\ \|\chi_u\nabla^\times R(\lambda)\chi_v\| &\le Ce^{-m_\lambda|u-v|} \end{aligned} \tag{4.6}$$

*for any $u$, $v \in \mathbb{R}^3$. Here the norms in the left-hand side are the operator norms in $L_2(\mathbb{R}^3;\mathbb{C}^3)$.*

We now consider the sesquilinear form

$$Q[\varphi, w] := \langle\nabla^\times\varphi, \frac{1}{\varepsilon_0}\nabla^\times w\rangle - \lambda\langle\varphi, w\rangle$$

with the domain $H^1(\mathbb{R}^3;\mathbb{C}^3)$.

Let $\varphi := R(\lambda)\chi_x u$. Note that $\varphi$ belongs to the domain of the operator $M_0$.

Let $p = \max(2\mathrm{dist}(x, S_l), 1)$ and let $\xi_x(y)$ be a nonnegative smooth cutoff function that depends on $y_1$ only, is supported in $(x_1 - (p+1), x_1 + (p+1))$, and such that it is equal to 1 on $[x_1 - p, x_1 + p]$. We assume further that $\xi_x(y) \le 1$ and $|\nabla \xi_x(y)| \le C$ for some constant $C$ and all $x, y \in \mathbb{R}^3$. Note that $\xi_x u \in H^1(\mathbb{R}^3; \mathbb{C}^3)$. Using $w = \xi_x u$, one gets

$$Q[\varphi, \xi_x u] = \langle M_0 \varphi, \xi_x u\rangle - \langle \lambda\varphi, \xi_x u\rangle = \langle \chi_x u, \xi_x u\rangle = \|\chi_x u\|^2 .$$

Thus, the goal is to estimate $Q[\varphi, \xi_x u]$ from above. On the other hand, using the equality $Mu = \lambda u$ and integration by parts, one obtains

$$\begin{aligned} Q[\varphi, \xi_x u] &= \langle \nabla^\times \varphi, \varepsilon_0^{-1} \nabla^\times(\xi_x u)\rangle - \langle \varphi, \xi_x \lambda u\rangle \\ &= \langle \nabla^\times \varphi, \tilde{\epsilon} \nabla^\times (\xi_x u)\rangle + \langle \nabla^\times \varphi, \varepsilon^{-1} \nabla^\times(\xi_x u)\rangle - \langle \varphi, \xi_x M u\rangle \\ &= \langle \nabla^\times \varphi, \tilde{\epsilon} \nabla^\times (\xi_x u)\rangle + \langle \nabla^\times \varphi, \varepsilon^{-1} \nabla^\times(\xi_x u)\rangle - \langle \nabla^\times(\xi_x\varphi), \varepsilon^{-1}\nabla^\times u\rangle, \end{aligned} \tag{4.7}$$

where we used the notation

$$\tilde{\epsilon}(x) := \frac{1}{\varepsilon_0(x)} - \frac{1}{\varepsilon(x)}.$$

Notice that $\tilde{\epsilon}$ is supported inside the strip $\mathcal{S}_l$.

Using the identity $\nabla^\times(\xi u) = \xi \nabla^\times u + \nabla\xi \times u$, the first two terms in the last line of (4.7) can be combined to obtain

$$\begin{aligned} &\langle \nabla^\times\varphi, \tilde{\epsilon}(\xi_x \nabla^\times u + \nabla\xi_x \times u)\rangle + \langle \nabla^\times \varphi, \varepsilon^{-1}(\xi_x \nabla^\times u + \nabla \xi_x \times u)\rangle \\ &\quad = \langle \nabla^\times \varphi, \xi_x \tilde{\epsilon} \nabla^\times u\rangle + \langle \nabla^\times\varphi, (\tilde{\epsilon} + \varepsilon^{-1})\nabla\xi_x \times u\rangle + \langle \nabla^\times \varphi, \varepsilon^{-1}\xi_x \nabla^\times u\rangle. \end{aligned}$$

The term $-\langle \nabla^\times(\xi_x\varphi), \varepsilon^{-1}\nabla^\times u\rangle$ can be expanded to

$$-\langle \xi_x \nabla^\times \varphi, \varepsilon^{-1}\nabla^\times u\rangle - \langle \nabla\xi_x \times \varphi, \varepsilon^{-1}\nabla^\times u\rangle.$$

Combining the last two expressions, we get

$$\langle \nabla^\times \varphi, \xi_x \tilde{\epsilon}\nabla^\times u\rangle + \langle \nabla^\times \varphi, \varepsilon_0^{-1} \nabla \xi_x \times u\rangle - \langle \nabla\xi_x \times \varphi, \varepsilon^{-1}\nabla^\times u\rangle. \tag{4.8}$$

Our last task in proving the theorem is to estimate from above the terms in (4.8). Let $V = [x_1 - p - 1, x_1 + p + 1] \times l\Omega$. This is a compact domain that can be covered by the union of $p+1$ fixed size domains $V_j = [b_j, b_j + 2] \times l\Omega$ and which contains the support of $(\xi\tilde{\epsilon})$. Also note that $\operatorname{dist}(x, V_j) \ge \operatorname{dist}(x, S_l)$. Using Lemma 4.1 and (3.4), we get for any $0 < \eta < m_\lambda$

$$\begin{aligned} |\langle \nabla^\times \varphi, \xi_x \tilde{\epsilon} \nabla^\times u\rangle| &\le \|\chi_V \nabla^\times \varphi\| \, \|\xi_x \tilde{\epsilon} \nabla^\times u\| \\ &\le C \, \|\textstyle\sum_j \chi_{V_j} \nabla^\times R(\lambda)\chi_x u\| \|\textstyle\sum_j \chi_{V_j}\nabla^\times u\| \\ &\le C p^2 (|x_1| + p + 1)^{2N} e^{-m_\lambda \operatorname{dist}(x, \mathcal{S}_l)} \\ &\le C(|x_1| + 1)^{2N} e^{-(m_\lambda - \eta)\operatorname{dist}(x, \mathcal{S}_l)}. \end{aligned} \tag{4.9}$$

We used here that $p = \max(2\operatorname{dist}(x, \mathcal{S}_l), 1)$ and denoted different constants by $C$.

Let us move now to estimating the last term in (4.8). Denote by $a > 0$ a number such that shifts of $l\Omega$ by vectors $aj$ with $j \in \mathbb{Z}^2$ cover the whole space $\mathbb{R}^2$. We denote

$$W_j := ([x_1 - p - 1, x_1 - p] \cup [x_1 + p, x_1 + p + 1]) \times (l\Omega + aj) .$$

Then $W_j = W_0 + (0, aj)$. Notice that $W = \bigcup_j W_j$ covers $\operatorname{supp} \nabla\xi$ and $\operatorname{dist}(x, W_j) \ge C_1(p + |j|) - C_2$.

We are now ready to estimate the last term of (4.8) from above. We proceed as before, using the lemma, the polynomial growth of $u$, and uniform boundedness of $\nabla\xi_x$.

$$
\begin{aligned}
|\langle \nabla^\times \varphi, \varepsilon_0^{-1}\nabla\xi_x^\times u\rangle| &\le C\sum_j \|\chi_{W_j}u\|\|\chi_{W_j}\nabla^\times R(z)\chi_x u\| \\
&\le C\sum_j (|x_1|+p+|j|+1)^{2N}e^{-m_\lambda \operatorname{dist}(x,W_j)} \\
&\le C(|x_1|+p+1)^{2N}e^{-Cm_\lambda \operatorname{dist}(x,\mathcal{S}_l)}\sum_j (1+|j|)^{2N}e^{-Cm_\lambda|j|} \\
&\le C(|x_1|+1)^{2N}e^{-Cm_\lambda \operatorname{dist}(x,\mathcal{S}_l)}.
\end{aligned} \tag{4.10}
$$

The middle term in (4.8) is estimated analogously. Combining these estimates, we get

$$\|\chi_x u\|^2 = Q[\varphi,\xi_x u] \le C(1+|x_1|)^{2N}e^{-Cm_\lambda \operatorname{dist}(x,\mathcal{S}_l)}.$$

This finishes the proof of the theorem. □

**4.3. Proof of Theorem 3.5.** In this periodic situation, operator $M$ has a complete family of generalized eigenfunctions that do not grow in the $x_1$-direction of periodicity. Indeed, according to Bloch-Floquet theory [**20**, **25**], a generalized eigenfunction $u = u_\lambda$ of $M$ corresponding to $\lambda$ can be chosen as $\tilde{u}(x)e^{ikx_1}$, where $\tilde{u}(x)$ is periodic in the $x_1$-direction with period $a$ and $k = 2\pi/a$. Thus, $u$ satisfies (3.6). Then, repeating the previous proof, one comes up with the estimate (3.7). □

## 5. Remarks

(1) Theorem 3.2 provides sufficient conditions for the existence of a $\delta$-net of the defect spectrum inside a spectral gap. One wonders how much of the gap the guided mode spectrum can occupy. Can it fill the whole gap? If so, under what conditions? There seem to be no rigorous results available concerning these questions.

(2) Results of [**2**] on improved Combes-Thomas resolvent estimates show that the exponential decay constant $m_\lambda$, which clearly depends on the distance of the point $\lambda$ from the spectrum, behaves as $\sqrt{(\lambda-\alpha)(\beta-\lambda)}$ inside a gap $G = (\alpha,\beta)$.

(3) As has already being mentioned, in order to have the full right to call the discovered modes "guided", one needs to show that they do not correspond to point spectrum (i.e., to bound states). Here the most treatable case should be of a periodic medium with a linear defect aligned along one of the lattice vectors. In this situation one can apply the Floquet-Bloch theory with respect to the axial variable of the waveguide and hope to use standard techniques applied in the case of Schrödinger operators with periodic potential (e.g., [**5**, **13**, **20**, **25**]). This happens to be not an easy task. Even in the case of "hard wall" periodic waveguides, when waves are contained in a periodic waveguide by Dirichlet, Neumann, or more general boundary conditions, this problem is nontrivial. Although it has been considered for a rather long time [**6**, **20**], the first real advances are very recent [**11**, **12**, **14**, **28**–**30**]. The case of photonic crystal waveguides is more complex, due to the absence of complete confinement of the waves,

which exponentially decay into the bulk, but do not vanish completely. Apparently, the most recent work devoted to this issue assumes the bulk material to be homogeneous, with an embedded periodic guide [**10**].

## 6. Acknowledgements

The authors express their gratitude to H. Ammari, D. Dobson, P. Exner, F. Santosa, R. Shterenberg, A. Sobolev, and T. Suslina for discussions and information. The authors thank the reviewer for useful comments.

## References


[1] H. Ammari and F. Santosa, *Guided waves in a photonic bandgap structure with a line defect*, SIAM J. Appl. Math. **64** (2004), no. 6, 2018–2033.

[2] J. M. Barbaroux, J. M. Combes, and P. D. Hislop, *Localization near band edges for random Schrödinger operators*, Helv. Phys. Acta **70** (1997), 16–43.

[3] Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, AMS, Providence, RI, 1968.

[4] F. A. Berezin and M. Shubin, *The Schrödinger Equation*, Moskov. Gos. Univ., Moscow, 1983 (in Russian), extended English translation: Kluwer Academic Publishers, 1991.

[5] M. Sh. Birman and T. A. Suslina, *Periodic magnetic Hamiltonian with a variable metric. The problem of absolute continuity*, Algebra i Analiz **11** (1999), no. 2; English translation in St. Petersburg Math. J. **11** (2000), no. 2, 203–232.

[6] V. I. Derguzov, *On the discreteness of the spectrum of the periodic boundary-value problem associated with the study of periodic waveguides*, Siberian Math. J. **21** (1980), 664–672.

[7] A. Figotin and A. Klein, *Localization of classical waves I: Acoustic waves*, Commun. Math. Phys. **180** (1996), 439–482.

[8] A. Figotin and A. Klein, *Localization of classical waves II: Electromagnetic waves*, Commun. Math. Phys. **184** (1997), no. 2, 411–441.

[9] A. Figotin and A. Klein, *Localized classical waves created by defects*, J. Stat. Phys. **86** (1997), no. 1-2, 165–177.

[10] N. Filonov and F. Klopp, *Absolute continuity of spectrum of a Schrodinger operator with a potential that is periodic in some direction and decays in others*, Doc. Math. **9** (2004), 107–121.

[11] R. L. Frank, *On the scattering theory of the Laplacian with a periodic boundary condition. I. Existence of wave operators*, Doc. Math. **8** (2003), 547–565.

[12] R. L. Frank and R. G. Shterenberg, *On the scattering theory of the Laplacian with a periodic boundary condition. II. Additional channels of scattering*, Doc. Math. **9** (2004), 57–77.

[13] L. Friedlander, *On the spectrum of a class of second order periodic elliptic differential operators*, Commun. Math. Phys. **229** (2002), 49–55.

[14] L. Friedlander, *Absolute continuity of the spectra of periodic waveguides*, Contemporary Math. **339** (2002), 37–42.

[15] J. D. Jackson, *Classical Electrodynamics*, 3rd ed., Wiley, New York, NY, 1998.

[16] J. D. Joannopoulos, S. G. Johnson, R. D. Meade, and J. N. Winn, *Photonic Crystals, Molding the Flow of Light*, Princeton Univ. Press, 2nd edition, Princeton, NJ, 2008.

[17] S. John, *Strong localization of photons in certain disordered dielectric superlattices*, Phys. Rev. Lett. **58** (1987), 2486–2489.

[18] S. G. Johnson and J. D. Joannopoulos, *Photonic Crystals, the Road from Theory to Practice*, Kluwer Academic Publishers, Boston, 2002.

[19] A. Klein, A. Koines, and M. Seifert, *Generalized eigenfunctions for waves in homogeneous media*, J. Funct. Anal. **190** (2002), no. 1, 255–291.

[20] P. Kuchment, *Floquet Theory for Partial Differential Equations*, Birkhäuser Verlag, Basel, 1993.

[21] P. Kuchment, *The mathematics of photonics crystals*, Chapter 7 in *Mathematical Modeling in Optical Science*, G. Bao, L. Cowsar, and W. Masters (Editors), SIAM, Philadelphia, 2001, 207–272.

[22] P. Kuchment and B.-S. Ong, *On guided waves in photonic crystal waveguides*, in *Waves in Periodic and Random Media*, Contemp. Math. **339**, AMS, Providence, RI, 2002, 105–115.

[23] J. T. Londergan, J. P. Carini, and D. P. Murdock, *Binding and Scattering in Two-Dimensional Systems*, Springer Verlag, Berlin, 1999.
[24] Photonic & Sonic Band-Gap and Metamaterial Bibliography, http://phys.lsu.edu/~jdowling /pbgbib.html.
[25] M. Reed and B. Simon *Methods of Modern Mathematical Physics, IV: Analysis of Operators*, Academic Press, New York, 1978.
[26] K. Sakoda, *Optical Properties of Photonic Crystals*, Springer Verlag, Berlin, 2001.
[27] M. Skorobogatiy and J. Yang, *Fundamentals of Photonic Crystal Guiding*, Cambridge Univ. Press, 2009.
[28] A. V. Sobolev and J. Walthoe, *Absolute continuity in periodic waveguides*, Proc. London Math. Soc. **(3)** (2002), no. 85, 717–741.
[29] R. G. Shterenberg, *Schrödinger operator in a periodic waveguide on the plane and quasi-conformal mappings*, Zap. Nauchn. Sem. POMI **295** (2003), 204–243 (in Russian).
[30] R. G. Shterenberg and T. A. Suslina, *Absolute continuity of the spectrum of a magnetic Schrödinger operator in a two-dimensional periodic waveguide*, Algebra i Analiz **14** (2002), 159–206; English transl. in St. Petersburg Math. J. **14** (2003).
[31] E. Yablonovitch, *Inhibited spontaneous emission in solid-state physics and electronics*, Phys. Rev. Lett. **58** (1987), 2059–2062.
[32] F. Zolla, G. Renversez, A. Nicolet, B. Kuhlmey, S. Guenneau, D. Felbacq, *Foundations Of Photonic Crystal Fibres*, World Sci. Publ., 2005.

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368

*E-mail address*: kuchment@math.tamu.edu

Veritas, 10300 Town Park Drive, Houston, Texas 77072

*E-mail address*: beng.ong@cggveritas.com