Proceedings of Symposia in Applied Mathematics

Generalized Transforms of Radon Type and Their Applications

Peter Kuchment

ABSTRACT. These notes represent an extended version of the contents of the third lecture delivered at the AMS Short Course "Radon Transform and Applications to Inverse Problems" in Atlanta in January 2005. They contain a brief description of properties of some generalized Radon transforms arising in inverse problems. Here by generalized Radon transforms we mean transforms that involve integrations over curved surfaces and/or weighted integrations. Such transformations arise in many areas, e.g. in Single Photon Emission Tomography (SPECT), Electrical Impedance Tomography (EIT) thermoacoustic Tomography (TAT), and other areas.

1. Introduction

The notes by E. T. Quinto in this volume have already introduced the reader to the properties of the Radon transform and its role in inverse problems, in particular in computerized tomography. In this text we show that in some applications one has to work with weighted (*attenuated*) transforms of Radon type, where the lines (planes) of integration are equipped with certain weights that need to be incorporated into the transform. On the other hand, there are also important applied problems, where the data provides the values of the integrals of an unknown quantity over a family of curved manifolds (e.g., spheres) rather than lines or planes. These manifolds of integrations might be equipped with some weights as well. Such transforms have been studied in rather general situation (e.g., [**19, 21, 22, 33, 34, 39, 51, 52, 54, 55, 58, 59, 60, 119, 120, 123, 124, 125, 128, 129, 130, 135**] and references therein), but a richer theory can be developed for more specific examples. As it often happens, transforms arising in applications, have a special structure that allows for a deep and beautiful analytic theory.

Although this does not exhaust all situations that fall under our topic, we will restrict ourselves to the following three (probably the most prominent) areas: Thermoacoustic Tomography (TAT), where integrations over spheres are involved, Single Photon Emission Computed Tomography (SPECT), where weighted transforms arise, and Electrical Impedance Tomography (EIT), where hyperbolic Radon transforms appear naturally.

In these notes we are unable to provide a comprehensive bibliography (which would take at least as much space as the whole notes). Apologies are extended to the authors whose work should have been, but was not mentioned explicitly.

2. Thermoacoustic tomography and the circular Radon transform

Tomographic methods of medical imaging, as well as of industrial non-destructive evaluation and geological prospecting are based on the following general procedure: one sends towards a non-transparent body some kind of a signal (acoustic or electromagnetic wave, X-ray, visual light photons, etc.) and measures the wave after it passes through the body. Then one tries to use the measured information to recover the internal structure of the object of study. The common feature of most traditional methods of tomography is that the same kinds of physical signals are sent and measured. Each of the methods has its own drawbacks. For instance, sometimes when imaging biological tissues, microwaves and optical imaging might provide good contrasts between different types of tissues, but are inferior in terms of resolution in comparison with ultrasound or X-rays. This, in particular, is responsible for the common low resolution of optical or electrical impedance tomography. On the other hand, ultrasound, while giving good resolution, often does not do a good job in terms of contrast. One of the recent trends is to combine different types of waves in a single imaging process. The best developed example is probably the thermoacoustic tomography (TAT or TCT) and its sibling photoacoustic tomography (PAT) (e.g., [**76**],[**148**]-[**151**]). In TAT, a short microwave pulse is sent through a biological object. At each internal location x certain energy $H(x)$ is absorbed. It is known, that cancerous cells often absorb several times more microwave (or radio frequency) energy than the normal ones, which means that significant contrast is expected between the values of $H(x)$ at tumorous and healthy locations. The absorbed energy causes a thermoelastic expansion, which in turn creates a pressure wave. This wave can be detected by ultrasound transducers placed at the edges of the object. Now the former weakness of ultrasound (low contrast) becomes an advantage. Indeed, in many cases (e.g., for mammography) one can assume the sound speed to be approximately constant. Hence, the sound waves detected by a transducer at any moment t of time are coming from points at a constant distance (depending on time t of travel and the sound speed) from its location. The strength of the signal coming from a location x reflects the energy absorption $H(x)$. Thus, one effectively measures the integrals of $H(x)$ over all spheres centered at the transducers' locations. In other words, in order to reconstruct H (and thus find cancerous locations) one needs to invert a generalized Radon transform that provides the integrals of H over spheres centered at all available transducers' locations [**76**], [**148**]-[**151**]. This method amazingly combines advantages of two types of radiation, while avoiding their deficiencies.

This discussion motivates the study of the following "circular" Radon transform[1]. Let $f(x)$ be a continuous function on $\mathbb{R}^n$, $n \geq 2$. We define its circular Radon transform as

$$Rf(p,r) = \int_{|y-p|=r} f(y) d\sigma(y),$$

where $d\sigma(y)$ is the surface area on the sphere $|y-p| = r$ centered at $p \in \mathbb{R}^n$.

The mapping from f to Rf is overdetermined, since the dimension of pairs (p,r) is $n+1$, while the function f depends on n variables only. This suggests to restrict the centers p to a set (hypersurface) $S \subset \mathbb{R}^n$, while not imposing any restrictions on the radii[2]. This restricted transform will be denoted by R_S:

$$R_S f(p,r) = Rf(p,r)|_{p \in S}.$$

Among central problems that naturally arise are:

• **Uniqueness of reconstruction**: is the information collected for a given set S of centers sufficient for the unique determination of the function f? In other words, is the operator R_S injective (on a specific function space)?

[1]Numerous other reasons to study this transform are known, e.g. Radar and Sonar imaging, approximation theory, PDEs, potential theory, complex analysis, etc. [**2, 93**]. Although in dimensions higher than two one should probably use the word "spherical" rather than "circular," we will use for simplicity the latter. This should not create any confusion.

[2]The most popular in TAT geometries of these surfaces (curves) S of centers (transducers) are spheres, planes, and cylinders [**148**]-[**150**].

• **Inversion formulas** and algorithms for R_S.
• **Stability** of the reconstruction.
• Description of the **range** of the transform: what conditions must ideal data satisfy?

All these questions have been resolved for the classical Radon transform [**66, 102, 104**]. However, they are more complex and not too well understood for the circular Radon transform.

We will now provide a brief survey of the known results and approaches to the problems listed above.

2.1. Injectivity. Here one is interested in finding which sets S guarantee uniqueness of reconstruction of a function f from its transform R_S. We introduce the following

DEFINITION 1. *The circular transform R is said to be injective on a set S (S is a* **set of injectivity***) if for any compactly supported continuous function f on $\mathbb{R}^n$ the condition $Rf(p,r) = 0$ for all $r \geq 0$ and all $p \in S$ implies $f \equiv 0$. In other words, S is a set of injectivity, if the mapping R_S is injective on $C_c(\mathbb{R}^n)$.*

The condition of compactness of support on f is essential in what follows. The situation is significantly different and much harder to study without compactness of support (or at least a sufficiently fast decay) [**1, 2**]. Fortunately, tomographic problems usually yield compactly supported functions.

One now arrives to the
Problem: Describe all sets of injectivity for the circular Radon transform R on $C_c(\mathbb{R}^n)$. In other words, we are looking for a description of those sets of positions of transducers that enable one to recover uniquely the energy deposition function.

This problem has been around in different guises for quite a while (e.g., [**2, 39, 90, 91**] and references therein). One of its most useful reformulations is the following: finding all possible nodal sets of oscillating infinite membranes. Namely, consider the initial value problem for the wave equation in $\mathbb{R}^n$:

$$u_{tt} - \triangle u = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R} \tag{1}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = f. \tag{2}$$

Then

$$u(x,t) = \frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}}\int_0^t r(t^2-r^2)^{(n-3)/2}(Rf)(x,r)dr, \quad t \geq 0.$$

Hence, it is not hard to show [**2**] that the original problem is equivalent to the problem of recovering $u_t(x,0)$ from the value of $u(x,t)$ on subsets of $S \times (-\infty, \infty)$.

Lemma 2. [**2**] *A set S is a non-injectivity set for $C_c(\mathbb{R}^n)$ if and only if there exists a non-zero compactly supported continuous function f such that the solution $u(x,t)$ of the problem (1)-(2) vanishes for any $x \in S$ and any t.*

In other words, injectivity sets are those for which the motion of the membrane over S gives complete information about the motion of the whole membrane. An analogous relation holds also for solutions of the heat equation [**2**].

So, what could the injectivity sets be? As it turns out, they are more common than the non-injectivity ones. So, one should better try to describe the "bad" (i.e., non-injectivity) sets, i.e. sets of transducers' positions from which one cannot recover the energy absorption function.

A simple example of a non-injectivity surface is any hyperplane S. Indeed, if f is odd with respect to this plane, then clearly $R_S f = 0$, so one cannot recover f from the data. It is known that in this case the odd functions are the only ones "eliminated" by R_S [**37, 73**]. In particular, any line on the plane is a non-injectivity set. There are other options as well. Let us consider for any $N \in \mathbb{N}$ the Coxeter system Σ_N of N lines $L_0, \ldots, L_{n-1}$ in the plane passing through the origin and forming equal angles π/N: $L_k = \{te^{i\pi k/n} | -\infty < t < \infty\}$. There exist non-zero compactly supported functions that are simultaneously odd with respect to all lines Σ_N (look at the Fourier series expansion with respect to the polar angle). So, Σ_N is a non-injectivity set as well. Any rigid motion ω preserves non-injectivity property, so $\omega\Sigma_N$ is also a non-injectivity set. It is not that obvious, but still not hard to prove that adding a finite set F does not change this property. The following remarkable theorem was conjectured by V. Lin and A. Pincus [**90, 91**] and proven by M. Agranovsky and E. Quinto [**2**]:

Theorem 3. [**2**] *A set $S \subset \mathbb{R}^2$ is an injectivity set for the circular Radon transform on $C_c(\mathbb{R}^2)$, if and only if it is not contained in any set of the form $\omega(\Sigma_N) \bigcup F$, where ω is a rigid motion in the plane and F is a finite set.*

The beautiful proof of this theorem in [**2**] is based on microlocal analysis and geometric properties of zeros of harmonic polynomials[3]. There are, however, some comments about non-injectivity sets that can be made without heavy techniques being involved.

The first important observation concerning non-injectivity sets is that they must be algebraic (i.e., sets of zeros of non-zero polynomials). In fact, if $R_S f = 0$ and f decays faster than any power, it is not hard to see that the following polynomials vanish on S: $Q_k(x) = \int_{\mathbb{R}^n} \|x - \xi\|^{2k} f(\xi) d\xi$. One might wonder whether they could all vanish identically and thus imply nothing about S. It is easy to prove that this cannot happen if the function decays exponentially.

Now applying Laplace operator one readily concludes that the lowest degree polynomial among Q_k is in fact harmonic:

LEMMA 4. *Let $f \in C_c(\mathbb{R}^n)$, and $P = Q_{k_0}[f]$ be the minimal degree nontrivial polynomial among Q_k, then P is harmonic and vanishes on S .*

Thus, if R is not injective on S, then S is the zero set of a harmonic polynomial.

COROLLARY 5. *Any set $S \subset \mathbb{R}^n$ of uniqueness for the harmonic polynomials is a set of injectivity for the transform R. E.g., if $U \subset \mathbb{R}^n$ is any bounded domain, then $S = \partial U$ is an injectivity set of R.*

This corollary is already good enough for many practical applications. Indeed, in one of the common practical set-ups one places transducers around a sphere S. The corollary guarantees uniqueness of reconstruction. In fact, algebraicity implies that the data over any open piece of the sphere has as much information as the data collected over the whole sphere, and thus also guarantees uniqueness (albeit at the price of significantly reduced stability of reconstruction [**93, 151**]).

The conjecture that describes non-injectivity sets in higher dimensions (still for compactly supported functions) is:

CONJECTURE 6. [**2**] *A set $S \subset \mathbb{R}^n$ is an injectivity set for the circular Radon transform on $C_c(\mathbb{R}^n)$, if and only if it is not contained in any set of the form $\omega(\Sigma) \bigcup F$, where ω is a rigid motion of $\mathbb{R}^n$, Σ is the zero set of a non-zero* **homogeneous** *harmonic polynomial, and F is an algebraic subset in $\mathbb{R}^n$ of co-dimension at least 2.*

[3]Albeit some simpler approaches have started to appear, e.g. in [**46, 5**], there is still no alternative proof of this theorem available, except some partial solutions in [**5**].

For $n = 2$, this boils down to Theorem 3.

The uniqueness problem remains unresolved in dimensions higher than 2, and even in dimension 2 it is not resolved for functions that are not compactly supported (albeit possibly very fast decaying). For instance, there is a belief that the statement of Theorem 3 should hold true for functions that decay fast, say super-exponentially. So far no-one has succeeded in proving this. In [**1**], a limited scope question was posed: does the claim of Corollary 5 concerning the boundaries of bounded domains hold true for functions from $L^p(\mathbb{R}^n), p < \infty$? The answer, given by the following theorem, shows that the situation is non-trivial:

THEOREM 7. [**1**] *The boundary S of any bounded domain in $\mathbb{R}^n$ is uniqueness set for $f \in L^p(\mathbb{R}^n)$ if $p \leq \frac{2n}{n-1}$. This is not true when $p > \frac{2n}{n-1}$, where spheres provide counterexamples.*

A new approach based on the wave equation interpretation that we have mentioned above and which promises possible new advances in this problem, is introduced in [**46**] (see also its further development in [**5**]; an early indication of this technique can be found in [**1**]). It uses essentially only the finite speed of propagation and domain of dependence for the wave equation. It boils down to the following idea: one has a free infinite oscillating membrane, but a (non-injectivity) set S stays fixed (nodal). Thus, one can also adopt a point of view that the membrane is just fixed along S. In this interpretation, the waves have to bypass S, while on the other hand the membrane is free and the waves are free to go without any obstacle. This sometimes gives a contradiction between two times of arrival, which in turn eliminates some sets S as possible non-injectivity sets.

2.2. Inversion. When S is a uniqueness set (e.g., a sphere) one is interested in reconstruction formulas for recovery of f from its transform $R_S f$. There are very few examples when such a formula is known, e.g. when S is a plane. Although in this case, as we know, there is no uniqueness, functions that are even with respect to S, or functions that are supported on one side of S can be reconstructed (e.g., [**6, 38, 51, 53, 104, 116**]). For the most interesting for TAT case of S being a sphere, inversion algorithms using special functions expansions are known (see [**6, 38, 42, 51, 104, 105, 109, 116**] and references therein concerning all these inversion formulas). However, analytic formulas (e.g., of backprojection type similar to the ones for the standard Radon transform) had not been known until recently. In [**46**], such explicit formulas were derived for odd dimensions under the condition that the unknown function is supported inside of the sphere S (which is not a restriction for TAT). The $3D$ version of one of the results of [**46**] is presented below:

THEOREM 8. *Let f be a smooth function supported in the unit ball centered at the origin in $\mathbb{R}^3$. Then for any x in this ball, the following reconstruction formulas hold true:*

$$\begin{aligned} f(x) &= -\frac{1}{8\pi^2} \int\limits_{|p|=1} \frac{1}{|x-p|} \frac{\partial^2 R_S}{\partial r^2}(p, |x-p|)dp \\ &= -\frac{1}{8\pi^2} \Delta_x \int\limits_{|p|=1} \frac{1}{|x-p|} R_S(p, |x-p|)dp. \end{aligned} \tag{3}$$

Here the set of centers S is the unit sphere centered at the origin.

Notice that if the function is not supported inside the unit ball, the formula would give its incorrect values even inside the ball.

Such formulas for $2D$ and higher even dimensions are still not known. However, it is easy to write approximate formulas (parametrices) either by using ideas of microlocal analysis in the spirit of [**19, 82, 106**] or just by mimicking the Radon case. Microlocal analysis of such formulas usually shows that they recover the singularities of the function correctly, albeit they do not reconstruct the values of f precisely. Simple iterative correction procedures significantly improve the quality of reconstruction and seem to provide reconstructions adequate for practical purposes (e.g., [**148**]-[**151**]).

One should also mention an important analytic tool, unfortunately not that well known in the applied community, the so called κ-operator developed in I. Gelfand's school (one can find its description for instance in [**50, 51**]). It provides a unified approach to inverting various Radon type transforms.

2.3. Stability of reconstruction. The microlocal analysis (i.e., in terms of wave front sets) approach, similar to the one used for singularity detection in the Radon transform (see the lectures by E. T. Quinto in this volume), provides the general answer of what can and what cannot be stably reconstructed. Notice that uniqueness results by themselves do not guarantee stability. For instance, as we have mentioned before, a small portion of the sphere covered by transducers guarantees uniqueness of reconstruction. However, most of the sharp details will disappear, since their reconstruction is unstable. Namely, the following rule describes in general the situation. If at each point of the object to be reconstructed and for each line passing through this point there is a transducer located on this line, then reconstruction of the object can be made as stable as from the regular Radon transform data. However, if there is a line through a point that does not pass through any transducer's location, then any possible boundary between tissues at this point that is normal to the line, will be blurred in the reconstructed image. One can find further details for the case of the standard Radon (or X-ray) transform in [**127**], for attenuated transforms in [**74, 75, 82**], and for circular transforms in [**93, 151**].

2.4. Range description. Knowing the range of the transform R_S could be very useful. For instance, the range theorems for the Radon transform have been used to correct errors in measured data, to complete incomplete data, and for other purposes. There is no such result obtained yet for the circular transform. The paper [**117**] contains a series of range conditions for the case of S being a sphere, albeit it is unknown yet whether this set of conditions is complete. As for the standard Radon transform, the conditions found in [**117**] are not hard to derive, it is their completeness that still is not known. Indeed, let S be the unit sphere in $\mathbb{R}^n$ centered at the origin and we know the function $g(p,r) = \int\limits_{|x-p|=r} f(x)dx$ for any $p \in S$. Then for any natural k we immediately conclude that the momentum

$$Q_k(p) = \int\limits_0^\infty r^{2k} g(p,r)dr = \int\limits_{\mathbb{R}^n} (|x|^2 - 2x\cdot p + |p|^2)^k f(x)dx, \tag{4}$$

viewed on the unit sphere, is a polynomial of degree k. This gives us a series of necessary conditions.

2.5. Implementation. We finish this section with examples of reconstructions from synthesized, as well as real data. These results and figures are borrowed from [**151**].

Fig. 1 shows a mathematical phantom that was used for reconstructions shown in the next picture.

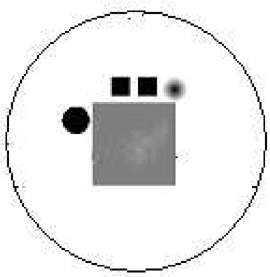

FIGURE 1. A mathematical phantom

Fig. 2 shows reconstructions of the phantom using different amounts of data in different columns. Namely, the detectors were placed correspondingly along an arc of approximately 90 degrees in the first quadrant, an arc containing two first quadrants, and finally a 360 degrees arc. The blurred parts of the boundaries are due to the limited view, which agrees with the microlocal analysis of the problem (see the discussion in the stability sub-section). Namely, a part of the boundary is blurred when its normals do not contain any detector locations. One can see how the existence of blurred parts depends on the detector arc. Different rows represent different reconstruction methods (see details in [**151**]).

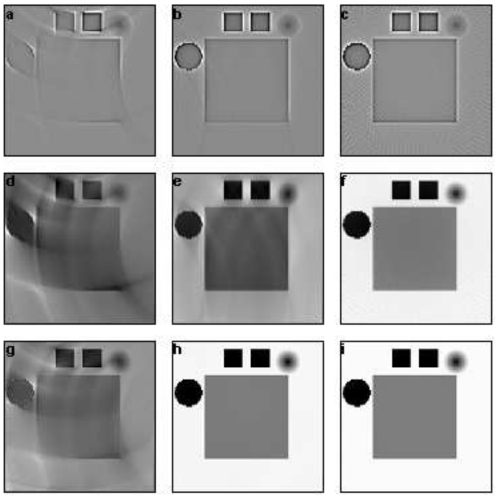

FIGURE 2. TAT reconstructions from the phantom data

In Fig. 3 one can see the photograph of a physical phantom (a piece of meat immersed into fat) and its reconstructions from the experimentally measured TAT data (measurements were performed in Prof. L. Wang's Optical Imaging Lab at Texas A& M University). We show TAT reconstructions that used limited data (left to right: detection arcs of 92 degrees, 202 degrees, and 360 degrees). The blurred parts of the boundaries again behave according to the theory.

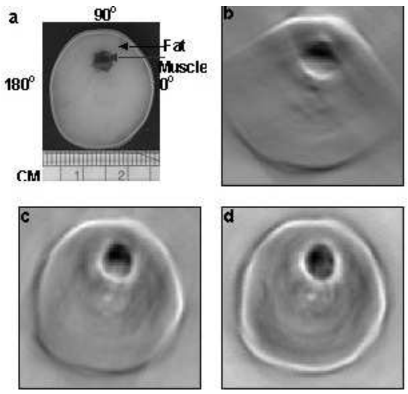

FIGURE 3. TAT reconstructions from experimental data

3. Emission tomography and attenuated Radon transform

Emission tomography deals with imaging of self-radiating bodies (as opposite to transmission imaging methods, where the source of radiation is outside of the object to be imaged). Let us describe briefly the main principle of the so called Single Photon Emission Computed Tomography (SPECT), a popular method of medical diagnostics (one can find more details in [**28, 68, 102, 104**])[4]. In SPECT, a patient is given a medication labelled by a radionuclide. The resulting emission is observed outside the body by collimated detectors that allow in only narrow beams of radiation (see Fig. 4). The goal is by measuring

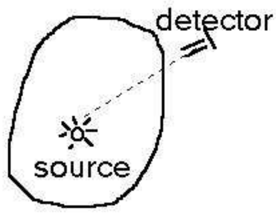

FIGURE 4. Single Photon Emission Computed Tomography

[4]We will consider here the $2D$ version, i.e. only the beams that belong to a specific plane will be taken into account. There has been a recent activity in $3D$ SPECT reconstructions that do not reduce to layer-by-layer $2D$ procedures, e.g. [**86, 107, 147**].

the intensity of the outgoing radiation to **reconstruct the interior intensity distribution** $f(x)$ of the radiation sources[5]. Let $\mu(x)$ be the **linear attenuation coefficient** (or just **attenuation**) of the body at the location x. Due to the absorption, a beam passing through the body, suffers losses. Assuming that effects of scatter are small, one can solve a simple transport equation to conclude (e.g., [**102**]) that the total detected intensity along a beam (straight line) L reaching the detector is

$$T_\mu f(L) = \int_L f(x) e^{-\int_{L_x} \mu(y)dy} dx. \tag{5}$$

Here L_x is the segment of the line L between the emission point x and the detector and dy denotes the standard linear measure on L. The operator T_μ is said to be the **attenuated Radon transform** with attenuation $\mu(x)$ of the function $f(x)$. One can make the formula above a little more specific by parametrizing the lines. Namely, let ω be the unit vector normal to the direction of L, $\omega^\perp$ be its 90^o degrees counterclockwise rotation, and s be the signed distance from the origin to L. Then the line L consists of points $s\omega + t\omega^\perp$, $t \in \mathbb{R}$. Now

$$T_\mu f(\omega, s) = \int_{-\infty}^{\infty} f(s\omega + t\omega^\perp) e^{-\int_t^\infty \mu(s\omega + \tau\omega^\perp) d\tau} dt. \tag{6}$$

In contrast with the standard Radon transform $f \to \int_L f(x)dx$, the resulting function depends on the orientation of the line L [6].

It is often assumed that the attenuation μ is constant inside the body and zero outside. If the body is convex and of a known shape, then it is easy to check (this was discovered first in [**144**]) that by a multiplication by a known function the attenuated transform can be reduced to a simpler one, called the **exponential Radon transform** of function f:

$$R_\mu f(\omega, s) = \int_{-\infty}^{\infty} f(s\omega + t\omega^\perp) e^{\mu t} dt \tag{7}$$

(here μ is constant). For this integral to make sense, the function $f(x)$ needs to have exponential decay at infinity sufficient to offset the effect of the exponential weight in the integral.

As before, the natural questions to ask about these two transforms are:

---

[5] This problem arises not only in medical imaging, but everywhere where one wants to reconstruct the interior of a self-radiating object, e.g. nuclear reactor, jet engine, etc. [**121, 122**].

[6] In practice one often averages over the two orientations, thus arriving to a function of non-oriented lines.

• **Injectivity**: Can a function of a natural class be reconstructed from its attenuated or exponential transforms?
• **Inversion formulae**, if injectivity is established.
• **Range**. Judging by the precedent of the standard Radon transform, these operators are unlikely to be surjective in any natural function spaces. So, what are the conditions the functions from the ranges of these operators must satisfy? As has been mentioned before, such knowledge is important for applications, as well as for understanding the analytic properties of these transforms.
• **Stability of inversion**.
• **Simultaneous reconstruction of the sources density f and attenuation μ.** This is an unusual question, which does not arise for the standard Radon transforms. In most cases not only the value of the intensity distribution $f(x)$, but the attenuation coefficient $\mu(x)$ as well is unknown. So, the question is whether it is possible to extract any information about both functions from the values of $T_\mu f$ or $R_\mu f$?

These problems will be briefly addressed below. In all cases we will describe first what is the situation with the simpler exponential transform, and then address the attenuated one. As it was mentioned, we will deal almost entirely with the $2D$ case.

3.1. Uniqueness of reconstruction and inversion formulae.

3.1.1. *Exponential transform.* One of the useful properties of the exponential transform R_μ is an analog of the projection-slice (also called Fourier-slice) theorem known for the Radon case. Namely, if f is compactly supported (or sufficiently fast exponentially decaying) function on the plane, a straightforward computation of Fourier transform leads to the formula

$$\widehat{R_\mu f}(\omega,\sigma) = \sqrt{2\pi}\widehat{f}(\sigma\omega + i\mu\omega^\perp). \tag{8}$$

Here the hat on the left is the $1D$ Fourier transform with respect to s, while on the right it stands for the $2D$ Fourier transform. I.e., projection data $R_\mu f$ provides the values of the Fourier transform of the function f on the following surface in $\mathbf{C}^2$:

$$S_\mu = \left\{ z = \sigma\omega + i\mu\omega^\perp \mid \sigma \in \mathbf{R}, \omega \in \mathbf{S}^1 \right\} \subset \mathbf{C}^2. \tag{9}$$

This is an indication that one can try to use methods of functions of several complex variables. Results of many studies (e.g., [**3, 40, 45, 78, 79, 81, 95, 99, 113, 114**]) show that the relation between two theories is indeed very deep. For instance, Paley-Wiener theorems that guarantee analyticity of $\widehat{f}$, together with the simple claim that the surface S_μ is a uniqueness set for analytic functions, prove uniqueness of reconstruction (i.e., injectivity) for the exponential Radon transform.

The first explicit inversion formula for the exponential Radon transform in the plane was obtained in [**145**] (see also discussion in [**102**]). An inversion procedure was also provided in [**14**]. To describe the formula from [**145**], we introduce the dual exponential Radon transform (**exponential backprojection**) $R^\#_\mu$: applied to a function $g(\omega, s)$, it produces a function on the plane according to

$$\left(R^\#_\mu g\right)(x) = \int_{\mathbf{S}^1} e^{\mu x\cdot\omega^\perp} g(\omega, x\cdot\omega)d\omega. \tag{10}$$

Then a not very difficult calculation gives

$$R^\#_{-\mu} R_\mu f = \left(2\frac{\cosh \mu\,|x|}{|x|}\right) * f, \tag{11}$$

So, in order to reconstruct f, one needs to perform a de-convolution. Let

$$\zeta_\mu(\sigma) = \begin{cases} |\sigma| & \text{when } |\sigma| > |\mu| \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

and I^{-1}_μ (a generalized Riesz potential) be the Fourier multiplier by $\zeta_\mu(\sigma)$. Then the inversion formula from [**145**] reads as follows:

$$f = \frac{1}{4\pi} R^\#_{-\mu} I^{-1}_\mu R_\mu f. \tag{13}$$

There is, however, another way of looking at the inversion. Let us start with the standard Fourier inversion formula that involves integration of $\widehat{f}$ over $\mathbb{R}^2$:

$$f(x) = (2\pi)^{-1/2} \int\limits_{\mathbb{R}^2} \widehat{f}(\xi) e^{ix\cdot\xi} d\xi. \tag{14}$$

Rewriting it as

$$f(x) = (2\pi)^{-1/2} \int\limits_{\mathbb{R}^2} \widehat{f}(z) e^{i(z_1x_1+z_2x_2)} dz_1 \wedge dz_2, \tag{15}$$

one notices that the integration over $\mathbb{R}^2$ is done of a holomorphic differential 2-form (we use here exponential decay of f and thus analiticity of $\widehat{f}$). Since we know the values of $\widehat{f}$ on S_μ, the idea is to use Cauchy type argument to shift the integration from $\mathbb{R}^2$ to S_μ. This is not straightforward (since, in particular, the surface S_μ has a hole in it and thus is not homological to $\mathbb{R}^2$), but can be achieved [**40, 45, 81, 95, 136, 137**]. This, in particular, leads in [**81, 137**] to a variety of inversion formulas. In particular, it was mentioned in [**81**] as a peculiar remark that one can invert an "obviously useless" (since such media apparently do not exist) exponential transform with the attenuation μ depending on the direction vector ω. It has turned out recently though, that such inversion formulas are important for some $3D$ SPECT scanning geometries [**86, 147**]. One should also notice recent results on exact inversion of the exponential transforms with "half-view" 180 degrees data [**107, 108, 131, 132**] [7]. These can also be treated using the "useless" formula from [**81**].

3.1.2. *Attenuated transform.* Problems on uniqueness of reconstruction and inversion are much harder for the full attenuated transform T_μ (5) and have been resolved only very recently. Due to rather technical nature of these results, we will just try to give the reader a general idea of those, as well as main references.

The first results on uniqueness were the local ones. It was shown in [**96**] that if $\mu \in C^2$, then the transform (5) is injective on functions with a sufficiently small support. The idea is that when f is localized in a small neighborhood of a point, then the weight is almost constant on the support of the function, and thus the attenuated transform is very close to the usual Radon transform. Now injectivity follows just from simple operator perturbation argument (a bounded operator close to an injective semi-Fredholm one is injective). The next significant step was made in [**43**], where uniqueness was established under the condition that the diameter of the object was "not too large". This result was sufficient for many practical situations, e.g. in medical applications it restricts the diameter of an object to 35.8cm. The proof was nontrivial and involved energy estimates. A breakthrough came in recent brilliant works [**8, 110, 111**], where uniqueness was proven under some mild smoothness condition on the attenuation and with no support size restrictions.

A similar uniqueness problem for attenuated X-ray transform in $3D$ and higher dimensions happens to be trivial [**43**]. Indeed, let a compactly supported function be in the kernel of the transform. Taking into account only the rays that belong to a two-dimensional plane barely touching the support of the function, one deals with the small support $2D$ situation and hence can conclude that the function must be zero. This allows one to "eat away" the whole support and to conclude that the function is in fact equal to zero.

[7] The reader should recall at this point that, unlike for the Radon transform, the exponential transform data are different at opposite locations.

The problem of uniqueness was also considered for transforms with more general positive weights $w(x, \omega)$:

$$\int f(y + t\omega)w(y + t\omega, \omega)dt \tag{16}$$

(e.g., [**21, 123, 124, 126, 88**]), and uniqueness results of different types were established. However, there exists a famous counterexample due to J. Boman [**20**] that shows that the condition of infinite smoothness of the weight function w alone does not guarantee uniqueness.

Let us now discuss **inversion**. An explicit inversion formula was found in ([**110, 111**]), while a less explicit procedure was discovered earlier in [**8**] (see also [**23, 30, 44, 56, 84, 85, 103**] for different derivations and implementations). Both approaches of [**8, 110, 111**] look at the more fundamental transport equation rather than the attenuated Radon transform itself, in order to obtain inversion formulas and procedures. We are not able to address here the details of these very interesting and illuminating techniques (see the recent surveys [**23, 44**]). Instead, we will just provide one of the incarnations of the inversion formula.

Let us denote by H the standard Hilbert transform and by R the standard Radon transform on the plane. Then the inversion formula of [**110**] can be written as follows:

$$f(x) = -\frac{1}{4\pi} Re\, div \int\limits_{S^1} \omega e^{(\mathcal{D}\mu)(x,\omega^\perp)} \left(e^{-h} H e^h T_\mu f\right)(\omega, x\omega)d\omega, \tag{17}$$

where $h = \frac{1}{2}(Id + iH)R\mu$ and $\mathcal{D}\mu$ is the so called divergent beam transform

$$\mathcal{D}\mu(\omega, y + t\omega) = \int_t^\infty \mu(y + \tau\omega)d\tau,$$

This formula was implemented numerically in [**56, 84, 85, 103**].

3.2. Range conditions. As before, we start with the simpler case of the exponential transform, which still leads to interesting analysis.

3.2.1. *Exponential transform.* The first appearance of the range conditions for R_μ was in [**14, 145**]. Let $f(x)$ be a smooth and compactly supported function on the plane and $g(\omega, s, \mu)$ its exponential Radon transform with attenuation μ. Representing $\omega = (\cos\phi, \sin\phi)$ and expanding $g(\omega, s, \mu)$ into the Fourier series with respect to ϕ:

$$g(\omega, s, \mu) = \sum_l g_l(s, \mu)e^{il\phi},$$

one can establish necessary range conditions in terms of the Fourier transform $\widehat{g}_l(\sigma,\mu)$ of $g_l(s,\mu)$ with respect to s. It was observed in [**14, 145**] that the function

$$(\sigma+\mu)^l\widehat{g}_l(\sigma,\mu)$$

is even with respect to σ for any $l \in \mathbf{Z}$. It was shown in [**79, 80**] that this set of conditions is complete.

These range conditions do not have the usual momentum form. A complete momentum type set of conditions was also found in [**79, 80**]: if $g(\omega,s) = R_\mu f$ for some $f \in C_0^\infty(\mathbf{R}^2)$, then the following identity is satisfied for any odd natural n:

$$\begin{aligned} &\textstyle\sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} \frac{d}{d\phi} \circ \left(\frac{d}{d\phi} - i\right) \circ \ldots \\ &\circ \left(\frac{d}{d\phi} - (k-1)i\right) \textstyle\int_{-\infty}^{\infty} (\mu s)^{n-k} g(s,\omega)ds = 0. \end{aligned} \tag{18}$$

Here i is the imaginary unit, $\omega = (\cos\phi, \sin\phi)$, and $\circ$ denotes composition of differential operators.

The condition (18) is not very intuitive and has been interpreted in several different ways [**3, 4, 83, 113**]. Even checking its necessity happens to be interesting, since a direct calculation shows that it is equivalent to the following series of identities for the usual sine function $\sin\phi$: for any odd natural n

$$\begin{aligned} &\textstyle\sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{d}{d\phi} - \sin\phi\right) \circ \left(\frac{d}{d\phi} - \sin\phi + i\right) \circ \ldots \\ &\circ \left(\frac{d}{d\phi} - \sin\phi + (k-1)i\right) \sin^{n-k}\phi = 0. \end{aligned} \tag{19}$$

The reader might want to try to establish these identities directly [**79**].

These conditions have also been studied in terms of complex analysis [**3, 4, 113, 114**]. It was shown in particular that they are essentially equivalent to some Bernstein-Hartogs' type theorems on extension of separately analytic functions [**3, 4, 113, 114**]. One of the amazing incarnations of the theorem is the following: let f be a function defined outside a disk in $\mathbb{R}^2$ and such that its restriction to any tangent line to the disk extends to an entire function of one variable. Then function f extends from $\mathbb{R}^2$ to an entire function on $\mathbb{C}^2$ [**3, 113, 114**].

Range conditions for the exponential X-ray transforms in dimensions higher than two were obtained in [**4**]. A nice discussion can be found in [**113, 114**].

Let us mention briefly some applications of these range conditions. They have been used for detecting and correcting some data errors arising from hardware imperfection in SPECT [**118**] and for treatment of incomplete data problems [**94**]. Another interesting application is to radiation therapy planning, which deals with the operator dual to the exponential X-ray transform [**35, 36**]. Range conditions have proven to be useful in this area as well [**35, 36, 77**].

3.2.2. *Attenuated transform.* Even before the breakthrough in inverting the attenuated transform was achieved, an infinite (albeit still incomplete) set of range conditions was found [**102**]. We denote by H the Hilbert transform on the line:

$$Hp(x) = \frac{1}{\pi} v.p. \int_{-\infty}^{\infty} \frac{p(y)}{x-y} dy.$$

Here $v.p.$ denotes the principal value of the integral. Let f and μ belong to the Schwartz space $\mathcal{S}(\mathbf{R}^2)$. Then, for $k > m \geq 0$ integers, we have

$$\int_{-\infty}^{\infty} \int_{0}^{2\pi} s^m e^{\pm ik\phi + 0.5(I \pm iH)R\mu(\omega,s)} T_\mu f(\omega, s) d\phi ds = 0, \tag{20}$$

where $\omega = (\cos\phi, \sin\phi)$, I is the identity operator, and $R\mu$ is the Radon transform of μ.

These conditions have been used for the simultaneous recovery of the sources $f(x)$ and attenuation $\mu(x)$ (see [**99**]-[**101**] and discussion below).

The recent publication [**112**] contains a complete set of range conditions.

3.3. Recovery of attenuation. As we have discussed in the beginning, simultaneous recovery of the sources density $f(x)$ and of the attenuation $\mu(x)$ is an important applied issue. At the first glance, this problem looks hopeless: we are trying to recover two functions $f(x)$ and $\mu(x)$ of two variables with the data $g = T_\mu f$ being a single function of two variables. This counting argument would be persuasive only if the operator T_μ were close to a surjective one. However, we know that T_μ has an infinite dimensional cokernel. Thus, when μ changes, the range could in principle rotate so much that for different values of the attenuation the ranges would have zero (or a "very small") intersection. If this were true, then both f and μ would be recoverable or "almost recoverable".

In the simplest case of the exponential X-ray transform, this problem was resolved in [**70, 139, 140**] (see also [**7**]). The range conditions were used to show that unless the function $f(x)$ is radial, both f and μ can be uniquely determined.

Recovery of a variable attenuation is definitely much more difficult. As in the exponential case, the range theorems are used to this end. The range conditions (20) have been used in order to do so [**99, 100, 101**].

Papers [**25, 26, 8, 110, 111, 112**] contain some additional indications on what techniques might be employed for that purpose. This issue, however, has not yet been resolved in a satisfactory way.

3.4. Stability of reconstruction. Reconstruction using attenuated or exponential Radon transform data is more unstable than in the usual Radon case, due to the presence of exponentially growing factors in the direct transforms and backprojections (10). However, due to the infinite dimensionality of the co-kernel of the operator, one has a huge freedom in modifying inversion formulas. This freedom (in the exponential transform case) has been used to select the most stable inversion algorithms [**61, 138**]. This still needs to be done for the attenuated transform (see [**85**] for initial considerations).

3.5. Other questions. Here the author wants to briefly mention some other related developments.

Attenuated transforms with non-smooth attenuations were considered in [**74, 75**]. Such transforms arise naturally in medical imaging, since the attenuation coefficient has discontinuities along the tissue boundaries, which introduce artifacts into reconstruction. This effect was studied in the papers cited above.

Effects of non-perfect collimation of detectors were treated in [**78**].

Exponential Radon (rather than X-ray) transforms were studied in [**136, 137**].

An interesting "universal" transform that has no free parameters, but still incorporates all exponential X-ray transforms was introduced and studied in [**40**]. This transform has a lot of invariant structure built in. Its study in particular reveals relations between the F. John's ultra-hyperbolic equation and boundary $\bar{\partial}$-operators.

4. Electric impedance tomography and hyperbolic Radon transform

Electrical Impedance Tomography (EIT) is a promising and inexpensive method of medical diagnostics and of industrial nondestructive testing (e.g., [**11, 12, 13, 24, 31, 32, 133**]). Here one tries to recover the conductivity of the interior of an object (e.g., patient's lungs and heart). The information about the electric conductivity is very important for medical diagnostics; it is also vital for some electrical procedures, such as defibrillation; it might also provide a cheap nondestructive evaluation technology. Here is the idea of EIT: one places electrodes around an object, creates through them various current patterns, and measures the corresponding boundary voltage drop responses (Fig. 5). Experimental and theoretical studies related to

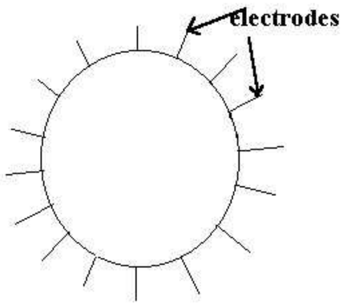

FIGURE 5. EIT

EIT are very active (see [**24, 31, 32, 133**] and references therein). Mathematically, the problem is much harder and less stable than the one of X-ray CT, or MRI. In particular, in most approaches no Radon type transform arises. We will address here only one direction, which does involve a generalized Radon transform, and surprisingly enough, a non-Euclidean one!

Let us describe first the mathematical formulation of EIT, which is the so called inverse conductivity problem in $2D$ (analogous formulations are available in higher dimensions as well). Let $U \subset \mathbb{R}^2$ be a sufficiently smooth domain (say, a disk) with boundary Γ. The unknown conductivity function $\beta(x)$ needs to be recovered from the following data. Given a known current function ψ on Γ, one measures the boundary value ϕ of the potential. Mathematically speaking, one solves the Neumann boundary value problem

$$\begin{cases} \nabla \cdot (\beta \nabla u) = 0 \quad \text{in} \quad U \\ \beta \frac{\partial u}{\partial \nu}\big|_{\Gamma} = \psi, \end{cases}$$

where ν is the unit outer normal vector on Γ and $\phi = u|_{\Gamma}$. All the pairs (ψ, ϕ) are assumed to be accessible. In other words, one knows the so called Dirichlet-to-Neumann operator $\Lambda_\beta : \phi \to \psi$. One needs to solve the nonlinear problem of recovery the conductivity β from this data. This happens to be a singularly hard inverse problem both analytically and numerically, not just (and not mainly) due to its nonlinearity, but mostly due to its severe instability. The main questions, as before are about uniqueness of reconstruction, its stability, and inversion procedures.

4.1. Uniqueness. After a long attempts and partial results, the uniqueness problem is essentially resolved positively (e.g., see [**9, 27, 72, 98, 143, 146**], and references therein), while the other questions (stability and reconstruction methods) are still under thorough investigation.

4.2. Stability. The general understanding is that the problem is highly unstable, so there is no hope to achieve the quality of reconstruction even close to the known for other common tomographic techniques. Indeed, as it will be in particular seen below, the problem is as unstable as the one of de-convolving a function with a Gaussian function. Saying this, we want to indicate that there are approaches that could possibly stabilize the problem. For instance, one could involve some additional available information about the image to be reconstructed, or one could try to reconstruct some useful functionals of the image rather than image's details, or finally one could try to change the physical set-up of EIT to improve stability of the reconstruction.

4.3. Reconstruction algorithms and the hyperbolic integral geometry. As we have already mentioned, the EIT problem (unlike the ones in X-ray, SPECT, PET, MRI, and TAT) is non-linear. Assuming that the unknown conductivity is a small variation of a constant, one can try to linearize the problem. This is exactly what the first practical algorithm of D. Barber and B. Brown [**11**]-[**13**] started with. Unfortunately, the linearized problem is still highly unstable. A study of this algorithm done in [**134**] lead in [**17, 18**] to the understanding that the linearized two-dimensional problem can be treated by means of hyperbolic geometry. Consider the $2D$ unit disk U. We can view U as the Poincare model of the hyperbolic plane $\mathbf{H}^2$ (e.g., [**15, 66**]). There are some indications why the hyperbolic geometry might be relevant for the inverse conductivity problem. Indeed, if one creates a dipole current through a point on the boundary of U, then the equipotential lines and the current lines form families of geodesics and horocycles in $\mathbf{H}^2$ (geodesics are the circular arcs orthogonal to the boundary of the unit disc, while horocycles are the circles tangential to that boundary). Besides, the Laplace equation that arises in the linearized problem is invariant with respect to the group of Möbius transformations, which serve as motions of the hyperbolic plane. It was discovered in [**17, 18**] (following analysis of [**134**]) that the linearized inverse conductivity problem in U reduces to the following problem on $\mathbf{H}^2$: the available data enables one to find the function $R_G(A * \beta)$, where R_G is the geodesic Radon transform on $\mathbf{H}^2$, A is an explicitly described radial function on $\mathbf{H}^2$:

$$A(r) = const(3\cosh^{-4} r - \cosh^{-2} r),$$

and the star $*$ denotes the (non-Euclidean) convolution on $\mathbf{H}^2$. Here the geodesic Radon transform integrates along geodesics in $\mathbf{H}^2$ with respect to the measure induced by the Riemannian metric on $\mathbf{H}^2$. Methods of harmonic analysis (Fourier and Radon transforms and their inversions) on the hyperbolic plane are well developed (e.g., [**16, 66, 67, 87, 89, 92**]). One hopes to use them to invert the geodesic Radon transform, to de-convolve, and as the result recover β. In particular, the papers mentioned above contain explicit inversion formulas for the hyperbolic geodesic Radon transform. The formula obtained in [**92**] was numerically implemented in [**48**] and works as nicely and stably as the standard inversions of the regular Radon transform[8].

As an illustration, we show in Fig. 6 below a numerical reconstruction from its geodesic Radon transform of a chessboard phantom.

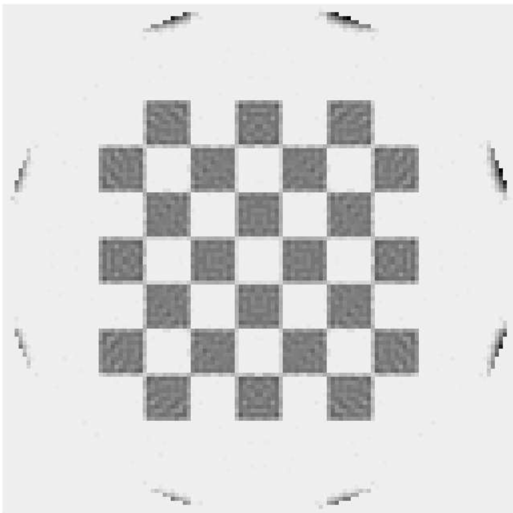

FIGURE 6. Hyperbolic reconstruction of a chessboard phantom.

The next Fig. 7 shows a similar reconstruction using a local tomography method (Λ-tomography, see the lectures by E. T. Quinto) that emphasizes boundaries.

So, hyperbolic Radon transforms can be computed and inverted numerically. However, as it is discussed above, the next step of the

[8]By an editorial error, all pictures have been omitted in [**48**]. They can be found at the URL http://www.math.tamu.edu/ kuchment/hypnum.pdf.

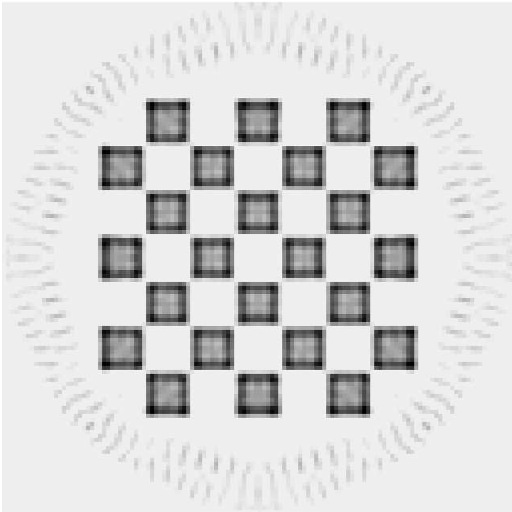

FIGURE 7. Local hyperbolic reconstruction of a chessboard phantom.

linearized EIT inversion needs to be de-convolution. Its numerical implementation can be attempted by using the well studied Fourier transform on the hyperbolic plane and its inversion [**66**]. The Fourier transform acts as follows:

$$f(z) \to Ff(\lambda, b) = \int_{\mathbf{H}} f(z) e^{(-i\lambda+1)\langle z,b\rangle} dm(z),$$

where $b \in \partial\mathbf{H}^2$, $\lambda \in \mathbf{R}$, $\langle z, b\rangle$ is the (signed) hyperbolic distance from the origin to the horocycle passing through the points z and b, and $dm(z)$ is the invariant measure on $\mathbf{H}^2$. The inverse Fourier transform is

$$g(\lambda, b) \to F^{-1}g(z) = const \int\int g(\lambda, b) e^{(i\lambda+1)\langle z,b\rangle} \lambda \tanh(\pi\lambda/2) d\lambda db.$$

These Fourier transforms were numerically implemented in [**48**]. However, the deconvolution part is the one that makes the whole problem extremely unstable. Indeed, to de-convolve, one needs to do the hyperbolic Fourier transform, to multiply it by an explicitly given function of exponential growth, and then to apply the inverse Fourier transform. Due to the exponential growth of the Fourier multiplier, such a procedure is extremely unstable and allows one to recover stably only very low frequencies, and hence to get a strongly blurred image only. So, it should not be possible to get sharp resolution EIT, unless some radical additional information is incorporated (e.g., some a priori knowledge about the image), or the physical set-up of the technique is changed.

At the first glance, the relation of the linearized inverse conductivity problem with the hyperbolic integral geometry does not seem to work in dimensions higher than two, due to lack of hyperbolic invariance of the governing equations. It was a surprise then, when it was shown in [**49**] that a combination of Euclidean and hyperbolic integral geometries still does the trick.

5. Acknowledgments

This work was partly supported by the NSF Grants DMS 9971674 and 0002195. The author thanks the NSF for this support. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

The author also expresses his gratitude to the reviewer for the most helpful remarks.

References

[1] M. Agranovsky, C. A. Berenstein, and P. Kuchment, Approximation by spherical waves in L^p-spaces, J. Geom. Anal., **6**(1996), no. 3, 365–383.

[2] M. L. Agranovsky, E. T. Quinto, Injectivity sets for the Radon transform over circles and complete systems of radial functions, J. Funct. Anal., **139** (1996), 383–413.

[3] V. Aguilar, L. Ehrenpreis, and P. Kuchment, Range conditions for the exponential Radon transform, J. d'Analyse Mathematique, 68(1996), 1-13.

[4] V. Aguilar and P. Kuchment, Range conditions for the multidimensional exponential X-ray transform, Inverse Problems 11(1995), 977-982.

[5] G. Ambartsoumian and P. Kuchment, On the injectivity of the circular Radon transform arising in thermoacoustic tomography, Inverse Problems **21** (2005), 473–485.

[6] L.-E. Andersson, On the determination of a function from spherical averages, SIAM J. Math. Anal. **19** (1988), no. 1, 214–232.

[7] Yu. E. Anikonov and I. Shneiberg, Radon transform with variable attenuation, Doklady Akad. Nauk SSSR, 316(1991), no.1, 93-95. English translation in Soviet. Math. Dokl.

[8] E.V. Arbuzov, A.L. Bukhgeim, and S.G. Kasantzev, Two-dimensional tomography problems and the theory of A-analytic functions, Siberian Adv. in Math. 8(1998), no.4, 1-20.

[9] K. Astala and L. Paivarinta, Calderon's inverse conductivity problem in the plane, ArXiv preprint math.AP/0401410, 2004.

[10] G. Bal, On the attenuated Radon transform with full and partial measurements, Inverse Problems **20** (2004), 399–418.

[11] D.C. Barber, B.H. Brown, Applied potential tomography, J. Phys. E.: Sci. Instrum. 17(1984), 723-733.

[12] D.C. Barber, B.H. Brown, Recent developments in applied potential tomography-APT, in: Information Processing in Medical Imaging, Nijhoff, Amsterdam, 1986, 106-121.

[13] D.C. Barber, B.H. Brown, Progress in electrical impedance tomography, in Inverse Problems in Partial Differential Equations, SIAM, 1990, pp. 151-164.

[14] S. Bellini, M. Piacentini, C. Caffario, and P. Rocca, Compensation of tissue absorption in emission tomography, IEEE Trans. ASSP 27(1979), 213-218.

[15] A. Berdon, The Geometry of Discrete Groups, Springer Verlag, New York-Heidelberg-Berlin, 1983.

[16] C. Berenstein, E. Casadio Tarabusi, Inversion formulas for the k-dimensional Radon transform in real hyperbolic spaces, Duke Math. J. 62(1991),no.3, 613-631.

[17] C. Berenstein, E. Casadio Tarabusi, The inverse conductivity problem and the hyperbolic X-ray transform, pp. 39-44 in [**54**].

[18] C. Berenstein, E. Casadio Tarabusi, Integral geometry in hyperbolic spaces and electrical impedance tomography, SIAM J. Appl. Math. 56(1996), 755-764.

[19] G. Beylkin, The inversion problem and applications of the generalized Radon transform, Comm. Pure Appl. Math. 37(1984), 579–599.

[20] J. Boman, An example of nonuniqueness for a generalized Radon transform. J. Anal. Math. 61(1993), 395–401.

[21] J. Boman and E. T. Quinto, Support theorems for real analytic Radon transforms, Duke Math. J. 55(1987), no.4, 943-948.

[22] J. Boman and E. T. Quinto, Support theorems for real analytic Radon transforms on line complexes, Trans. Amer. Math. Soc 335(1993), 877-890.

[23] J. Boman and J.-O. Strömberg, Novikovs inversion formula for the attenuated Radon transform a new approach, J. Geom. Anal. **14** (2004), no. 2, 185–198.

[24] L. Borcea, Electrical impedance tomography, Inverse Problems **18** (2002) R99–R136.

[25] A. V. Bronnikov, Numerical solution of the identification problem for the attenuated Radon transform, Inverse Problems, **15** (1999), no. 5, 1315–1324.

[26] A. V. Bronnikov and A. Kema, Reconstruction of attenuation map using discrete consistency conditions, IEEE Trans. Med. Imaging, **19** (2000), no. 5, 451–462.

[27] R. Brown and G. Uhlmann, Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions, Comm. Part. Dif and only if. Equat. **22** (1997), no. 5–6, 1009-1027.

[28] T.F. Budinger, G.T. Gullberg, and R.H. Huseman, Emission computed tomography, pp. 147-246 in [**68**].

[29] A.L. Bukhgeim, Inversion formulas in inverse problems, a supplement to M. M. Lavrent'ev and L. Ya. Savel'ev, Linear Operators and Ill-posed Problems, Translated from the Russian, Consultants Bureau, New York; "Nauka", Moscow, 1995.

[30] A.L. Bukhgeim and S. G. Kazantsev, The attenuated Radon transform inversion formula for divergent beam geometry, preprint, 2002.

[31] M. Cheney, D. Isaacson, and J.C. Newell, Electrical Impedance Tomography, SIAM Review, 41, No. 1, (1999), 85-101.

[32] B. Cipra, Shocking images from RPI, SIAM News, July 1994, 14-15.

[33] A. Cormack, The Radon transform on a family of curves in the plane, Proc. Amer. Math. Soc. **83** (1981), no. 2, 325–330.

[34] A. Cormack and E.T. Quinto, A Radon transform on spheres through the origin in $\mathbb{R}^n$ and applications to the darboux equation, Trans. Amer. Math. Soc. **260** (1986), no. 2, 575–581.

[35] A. Cormack and E.T. Quinto, A problem in radiotherapy: questions of non-negativity, Internat. J. Imaging Systems and Technology, 1(1989), 120–124.

[36] A. Cormack and E.T. Quinto, The mathematics and physics of radiation dose planning, Contemporary Math. 113(1990), 41-55.

[37] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volume II Partial Differential Equations*, Interscience, New York, 1962.

[38] A. Denisjuk, Integral geometry on the family of semi-spheres. Fract. Calc. Appl. Anal. **2**(1999), no. 1, 31–46.

[39] L. Ehrenpreis, *The Universality of the Radon Transform*, Oxford Univ. Press 2003.

[40] L. Ehrenpreis, P. Kuchment, and A. Panchenko, The exponential X-ray transform and Fritz John's equation. I. Range description, in *Analysis, Geometry, Number Theory: the Mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998)*, 173–188, Contemporary Math., 251, Amer. Math. Soc., Providence, RI, 2000.

[41] C. L. Epstein and. B. Kleiner, Spherical means in annular regions, Comm. Pure Appl. Math. **XLVI** (1993), 441–451.

[42] J. A. Fawcett, Inversion of n-dimensional spherical averages, SIAM J. Appl. Math. **45**(1985), no. 2, 336–341.

[43] D.V. Finch, Uniqueness for the X-ray transform in the physical range, Inverse Problems 2(1986), 197-203.

[44] D.V. Finch, The attenuated X–ray transform: recent developments, in *Inside Out : Inverse Problems and Applications* , G. Uhlmann (Editor), 47–66, Cambridge Univ. Press 2003.

[45] D.V. Finch and A. Hertle, The exponential Radon transform, Contemporary Math. 63(1987), 67-74.

[46] D. Finch, Rakesh, and S. Patch, Determining a function from its mean values over a family of spheres, SIAM J. Math. Anal. **35** (2004), no. 5, 1213–1240.

[47] L. Flatto, D. J. Newman, H. S. Shapiro, The level curves of harmonic functions, Trans. Amer. Math. Soc. **123** (1966), 425–436.

[48] B. Fridman, P. Kuchment, K. Lancaster, S. Lissianoi, D. Ma, M. Mogilevsky, V. Papanicolaou, and I. Ponomarev, Numerical harmonic analysis on the hyperbolic plane. Appl. Anal. 76(2000), no. 3-4, 351–362.

[49] B. Fridman, P. Kuchment, D. Ma, and V. Papanicolaou, Solution of the linearized conductivity problem in the half space via integral geometry, in *Voronezh Winter Mathematical Schools*, P. Kuchment (Editor), 85–95, Amer. Math. Soc. Transl. Ser. 2, 184, Amer. Math. Soc., Providence, RI, 1998.

[50] I. Gelfand, S. Gindikin, and M. Graev, Integral geometry in affine and projective spaces, J. Sov. Math. 18(1980), 39-167.

[51] I. Gelfand, S. Gindikin, and M. Graev, *Selected Topics in Integral Geometry*, Transl. Math. Monogr. v. 220, Amer. Math. Soc., Providence RI, 2003.

[52] S. Gindikin (Editor), *Applied Problems of the Radon Transform*, AMS, Providence, RI, 1994.

[53] S. Gindikin, Integral geometry on real quadrics, in *Lie groups and Lie algebras: E. B. Dynkin's Seminar*, 23–31, Amer. Math. Soc. Transl. Ser. 2, 169, Amer. Math. Soc., Providence, RI, 1995.

[54] S. Gindikin and P. Michor (Editors), *75 Years of the Radon Transform*, Internat. Press 1994

[55] A. Greenleaf and G. Uhlmann, Microlocal techniques in integral geometry, Contemporary Math. 113(1990), 149-155.

[56] J.-P. Guillement, F. Jauberteau, L. Kunyansky, R. Novikov, and R. Trebossen, On SPECT imaging based on an exact formula for the nonuniform atenuation correction, Inverse Problems, 18 (2002) pp. L11-L19.

[57] J.-P. Guillement and R. Novikov, A noise property analysis of single-photon emission computed tomography data, Inverse Problems **20** (2004), 175–198.

[58] V. Guillemin, Fourier integral operators from the Radon transform point of view, Proc. Symposia in Pure Math., 27(1975) 297-300.

[59] V. Guillemin and S. Sternberg *Geometric Asymptotics*, Amer. Math. Soc., Providence, RI, 1977.

[60] V. Guillemin, On some results of Gelfand in integral geometry Proc. Symposia in Pure Math., 43(1985) 149-155.

[61] W. G. Hawkins, P. K. Leichner, and N. C. Yang, The circular harmonic transform for SPECT reconstruction and boundary conditions on the Fourier transform of the sinogram, IEEE Trans. Med. Imag. **7** (1988), 135–148.
[62] I. Hazou and D. Solmon, Inversion of the exponential Radon transform I, Analysis, Math. Methods in Appl. Sci. 10(1988), 561-574.
[63] I. Hazou and D. Solmon, Inversion of the exponential Radon transform II, Numerics, Math. Methods Appl. Sci. 13(1990), no. 3, 205–218.
[64] I. Hazou and D. Solmon, Filtered-backprojection and the exponential Radon transform, Math. Anal. Appl. 141(1989), no.1, 109-119.
[65] U. Heike, Single-photon emission computed tomography by inverting the attenuated Radon transform with least-squares collocation, Inverse Problems 2(1986), 307-330.
[66] S. Helgason, *The Radon Transform*, Birkhäuser, Basel 1980.
[67] S. Helgason, The totally-geodesic Radon transform on constant curvature spaces, Contemporary Math. 113(1990), 141-149.
[68] G. Herman (Ed.), *Image Reconstruction from Projections* , Topics in Applied Physics, v. 32, Springer Verlag, Berlin, New York 1979.
[69] A. Hertle, On the injectivity of the attenuated Radon transform, Proc. Amer. Math. Soc. 92(1984), 201-205.
[70] A. Hertle, The identification problem for the constantly attenuated Radon transform, Math. Z. 197(1988), 13-19.
[71] D. Isaacson and M. Cheney, Current problems in impedance imaging, in *Inverse Problems in Partial Differential Equations*, SIAM, 1990, pp. 141-149.
[72] V. Isakov, *Inverse Problems for Partial Differential Equations*, Applied Mathematical Sciences, v. 127. Springer-Verlag, New York, 1998.
[73] F. John, *Plane Waves and Spherical Means, Applied to Partial Differential Equations*, Dover 1971.
[74] A. Katsevich, Local tomography for the generalized Radon transform, SIAM J. Appl. Math. 57(1997), 1128–1162.
[75] A. Katsevich, Local tomography with nonsmooth attenuation, Trans. Amer. Math. Soc. 351(1999), no. 5, 1947–1974.
[76] R. A. Kruger, P. Liu, Y. R. Fang, and C. R. Appledorn, Photoacoustic ultrasound (PAUS)reconstruction tomography, Med. Phys. **22** (1995), 1605-1609.
[77] P. Kuchment, On positivity problems for the Radon transform and some related transforms, Contemporary Math., 140(1993), 87-95.
[78] P. Kuchment, On inversion and range characterization of one transform arising in emission tomography, pp. 240-248 in [**54**].
[79] P. Kuchment and S. Lvin, Paley-Wiener theorem for exponential Radon transform, Acta Appl. Math. 18(1990), 251-260
[80] P. Kuchment and S. Lvin, The range of the exponential Radon transform, Soviet Math. Dokl. 42(1991), no.1, 183-184.
[81] P. Kuchment and I. Shneiberg, Some inversion formulas in the single photon emission tomography, Appl. Anal. 53(1994), 221-231.
[82] P. Kuchment, K. Lancaster, and L. Mogilevskaya, On local tomography, Inverse Problems, 11(1995), 571-589.
[83] P. Kuchment and E. T. Quinto, Some problems of integral geometry arising in tomography, chapter XI in [**39**].

[84] L. Kunyansky, A new SPECT reconstruction algorithm based on the Novikov's explicit inversion formula, Inverse Problems 17(2001), 293–306.

[85] L. Kunyansky, Analytic reconstruction algorithms in emission tomography with variable attenuation, J. of Computational Methods in Science and Engineering (JCMSE), 1 , Issue 2s-3s (2001), pp. 267-286.

[86] L. Kunyansky, Inversion of the 3D exponential parallel-beam transform and the Radon transform with angle-dependent attenuation, Inverse Problems **20** (2004) 1455–1478.

[87] A. Kurusa, The Radon transform on hyperbolic space, Geometriae Dedicata 40(1991), no.2, 325-339.

[88] A. Kurusa, The invertibility of the Radon transform on abstract rotational manifolds of real type, Math. Scand. 70(1992), 112-126.

[89] A. Kurusa, Support theorems for totally geodesic Radon transforms on constant curvature spaces, Proc. AMS 122(1994), no.2, 429-435

[90] V. Ya. Lin and A. Pinkus, Fundamentality of ridge functions, J. Approx. Theory, **75** (1993), 295–311.

[91] V. Ya. Lin and A. Pinkus, Approximation of multivariable functions, in *Advances in computational mathematics*, H. P. Dikshit and C. A. Micchelli, eds., World Sci. Publ., 1994, 1-9.

[92] S. Lissianoi and I. Ponomarev, On the inversion of the geodesic Radon transform on the hyperbolic plane, Inverse Problems 13(1977), 1-10.

[93] A. K. Louis and E. T. Quinto, Local tomographic methods in Sonar, in *Surveys on solution methods for inverse problems*, pp. 147-154, Springer, Vienna, 2000.

[94] S. Lvin, Data correction and restoration in emission tomography, pp. 149–155 in [**129**].

[95] A. Markoe, Fourier inversion of the attenuated X-ray transform, SIAM J. Math. Anal. 15(1984), no.4, 718-722.

[96] A. Markoe and E. T. Quinto, An elementary proof of local invertibility for generalized and attenuated Radon transforms. SIAM J. Math. Anal. 16(1985), no. 5, 1114–1119.

[97] C. Mennesier, F. Noo, R. Clackdoyle, G. Bal, and L. Desbat, Attenuation correction in SPECT using consistency conditions for the exponential ray transform, Phys. Med. Biol. **44** (1999), 2483–2510.

[98] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Ann. Math. 143(1996), 71-96.

[99] F. Natterer, On the inversion of the attenuated Radon transform, Numer. Math. 32(1979), 431-438.

[100] F. Natterer, Computerized tomography with unknown sources, SIAM J. Appl. Math. 43(1983), 1201-1212.

[101] F. Natterer, Exploiting the range of Radon transform in tomography, in: Deuflhard P. and Hairer E. (Eds.), Numerical treatment of inverse problems in differential and integral equations, Birkhäuser Verlag, Basel 1983.

[102] F. Natterer, *The mathematics of computerized tomography*, Wiley, New York, 1986.

[103] F. Natterer, Inversion of the attenuated Radon transform, Inverse Problems 17(2001), no. 1, 113–119.

[104] F. Natterer and F. Wübbeling, *Mathematical Methods in Image Reconstruction*, Monographs on Mathematical Modeling and Computation v. 5, SIAM, Philadelphia, PA 2001.

[105] S. Nilsson, Application of fast backprojection techniques for some inverse problems of integral geometry, Linkoeping studies in science and technology, Dissertation 499, Dept. of Mathematics, Linkoeping university, Linkoeping, Sweden 1997.

[106] C. J. Nolan and M. Cheney, Synthetic aperture inversion, Inverse Problems **18**(2002), 221–235.

[107] F. Noo, R. Clackdoyle, and J.–M. Wagner, Inversion of the $3D$ exponential X-ray transform for a half equatorial band and other semi-circular geometries, Phys. Med. Biol. **47** (2002), 2727–35.

[108] F. Noo and J.–M. Wagner, Image reconstruction in $2D$ SPECT with 180^O acquisition, Inverse Problems, 17(2001), 1357–1371.

[109] S. J. Norton, Reconstruction of a two-dimensional reflecting medium over a circular domain: exact solution, J. Acoust. Soc. Am. **67** (1980), 1266-1273.

[110] R. Novikov, Une formule d'inversion pour la transformation d'un rayonnement X attenue, C. R. Acad. Sci. Paris Ser. I Math 332(2001), no. 12, 1059–1063.

[111] R. Novikov, An inversion formula for the attenuated X-ray transformation, Ark. Math., 40(2002), 145–167.

[112] R. Novikov, On the range characterization for the two–dimensional attenuated X–ray transform, Inverse Problems 18(2002), 677–700.

[113] O. Öktem, Comparing range characterizations of the exponential Radon transform, Research reports in Math., no.17, 1996, Dept. Math., Stockholm Univ., Sweden.

[114] O. Öktem, Extension of separately analytic functions and applications to range characterization of the exponential Radon transform, in *Complex Analysis and Applications (Warsaw, 1997)*, Ann. Polon. Math. 70(1998), 195–213.

[115] V. P. Palamodov, An inversion method for an attenuated x-ray transform, Inverse Problems **12** (1996), 717–729.

[116] V. P. Palamodov, Reconstruction from limited data of arc means, J. Fourier Anal. Appl. **6** (2000), no. 1, 25–42.

[117] S. K. Patch, Thermoacoustic tomography - consistency conditions and the aprtial scan problem, Phys. Med. Biol. **49** (2004), 1–11.

[118] I. Ponomarev, Correction of emission tomography data. Effects of detector displacement and non-constant sensitivity, Inverse Problems, 10(1995) 1-8.

[119] D. A. Popov, The Generalized Radon Transform on the Plane, the Inverse Transform, and the Cavalieri Conditions, Funct. Anal. and Its Appl., **35** (2001), no. 4, 270–283.

[120] D. A. Popov, The Paley–Wiener Theorem for the Generalized Radon Transform on the Plane, Funct. Anal. and Its Appl., **37** (2003), no. 3, 215–220.

[121] N.G. Preobrazhensky and V.V. Pikalov, *Unstable Problems of Plasma Diagnostic*, Nauka, Novosibirsk, 1982.

[122] N.G. Preobrazhensky and V.V. Pikalov, *Reconstructive Tomography in Gas Dynamics and Plasma Physics*, Nauka, Novosibirsk, 1987.

[123] E. T. Quinto, The dependence of the generalized Radon transform on defining measures, Trans. Amer. Math. Soc. 257(1980), 331–346.

[124] E.T. Quinto, The invertibility of rotation invariant Radon transforms, J. Math. Anal. Appl., 91(1983), 510-522.

[125] E. T. Quinto, Null spaces and ranges for the classical and spherical Radon transforms, J. Math. Anal. Appl. **90** (1982), no. 2, 408–420.

[126] E. T. Quinto, Pompeiu transforms on geodesic spheres in real analytic manifolds, Israel J. Math. 84(1993), 353–363.

[127] E. T. Quinto, Singularities of the X-ray transform and limited data tomography in $\mathbf{R}^2$ and $\mathbf{R}^3$, SIAM J. Math. Anal. 24(1993), 1215–1225.

[128] E. T. Quinto, Radon transforms on curves in the plane, in *Tomography, Impedance Imaging,. and Integral Geometry*, 231–244, Lectures in Appl. Math., Vol. 30, Amer. Math. Soc. 1994.

[129] E.T. Quinto, M. Cheney, and P. Kuchment (Editors), *Tomography, Impedance Imaging, and Integral Geometry*, Lectures in Appl. Math., vol. 30, AMS, Providence, RI 1994.

[130] V. G. Romanov, Reconstructing functions from integrals over a family of curves, Sib. Mat. Zh. **7** (1967), 1206–1208.

[131] H. Rullgård, An explicit inversion formula for the exponential Radon transform using data from 180 degrees, Ark. Mat. **42** (2004), 353–362.

[132] H. Rullgård, Stability of the inverse problem for the attenuated Radon transform with 180 degrees data, Inverse problems **20** (2004), 781–797.

[133] F. Santosa, Inverse problem holds key to safe, continuous imaging, SIAM News, July 1994, 1 and 16-18.

[134] F. Santosa, M. Vogelius, A backprojection algorithm for electrical impedance imaging, SIAM J. Appl. Math., 50(1990), 216-241.

[135] V. A. Sharafutdinov, *Integral Geometry of Tensor Fields*, V.S.P. Intl Science 1994.

[136] I. Shneiberg, Exponential Radon transform, Doklady Akad. Nauk SSSR, 320(1991), no.3, 567-571. English translation in Soviet. Math. Dokl.

[137] I. Shneiberg, Exponential Radon transform, pp. 235-246 in [**52**].

[138] I. Shneiberg, I. Ponomarev, V. Dmitrichenko, and S. Kalashnikov, On a new reconstruction algorithm in emission tomography, in [**52**], 247–255.

[139] D. Solmon, Two inverse problems for the exponential Radon transform, in *Inverse Problems in Action,*(P.S. Sabatier, editor), 46-53, Springer Verlag, Berlin 1990.

[140] D. Solmon, The identification problem for the exponential Radon transform, Math. Methods in the Applied Sciences, 18(1995), 687-695.

[141] E. Somersalo, M. Cheney, D. Isaacson, E. Isaacson, Layer-stripping: a direct numerical method for impedance imaging, Inverse Problems, 7(1991), 899- 926.

[142] J. Sylvester, A Convergent Layer-Stripping Algorithm for the Radially Symmetric Impedance Tomography Problem, Comm. in Part. Dif and only if.Equat., 17(1992), 1955-1994.

[143] J. Sylvester and G. Uhlmann, The Dirichlet to Neumann map and its applications, in “Inverse Problems in Partial Differential Equations”, SIAM, 1990, pp. 101-139.

[144] O.J. Tretiak and P. Delaney, The exponential convolution algorithm for emission computed axial tomography, Proc. Symp. Appl. Math., vol. 27, Amer. Math. Soc., Providence, R.I. 1982, pp. 25–33.

[145] O.J. Tretiak and C. Metz, The exponential Radon transform, SIAM J.Appl.Math. 39(1980), 341-354.

[146] G. Uhlmann, Inverse boundary value problems and applications, Asterisque 207(1992), 153-211

[147] J.–M. Wagner, F. Noo, and R. Clackdoyle, Exact inversion of the exponential X-ray transform for rotating slant-hole (RSH) SPECT, Phys. Med. Biol. **47** (2002), 2713–26.

[148] M. Xu and L.-H. V. Wang, Time-domain reconstruction for thermoacoustic tomography in a spherical geometry, IEEE Trans. Med. Imag. **21** (2002), 814-822.

[149] Y. Xu, D. Feng, and L.-H. V. Wang, Exact frequency-domain reconstruction for thermoacoustic tomography: I. Planar geometry, IEEE Trans. Med. Imag. **21** (2002), 823-828.

[150] Y. Xu, M. Xu, and L.-H. V. Wang, Exact frequency-domain reconstruction for thermoacoustic tomography: II. Cylindrical geometry, IEEE Trans. Med. Imag. **21** (2002), 829-833.

[151] Y. Xu, L. Wang, G. Ambartsoumian, and P. Kuchment, Reconstructions in limited view thermoacoustic tomography, Medical Physics 31(4) April 2004, 724-733.

Mathematics Department, Texas A&M University, College Station, TX 77845

E-mail address: kuchment@math.tamu.edu