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Asymptotics of Spectra of Neumann Laplacians in Thin Domains

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ABSTRACT. We discuss convergence of spectra of Neumann Laplace operators on thin domains to spectra of appropriate differential operators on graphs.

1. Introduction

In this paper we study the behaviour of the spectrum of the Neumann Laplacian in a thin graph–like domain when the domain's width tends to zero, i.e. when the domain looks like a "fattened graph." Problems of this type arise naturally in many areas of physics and mathematics, most notably in theoretical studies of problems of mesoscopic physics, superconductivity, photonic crystals, and other areas. In order not to overload this paper, we refer the reader to the survey [**4**] for further motivation, references, and related results. The main observation is that generally speaking there usually is a differential (rather than difference) operator on the graph whose spectrum provides the limit of the spectrum of the original Neumann Laplacian. This statement is not precise, since first of all the limit depends on how in particular domain shrinks to the graph, and secondly the resulting operators might act on a bigger space than the natural L^2 space on the graph. All these distinctions will be spelled out in the sections below.

The proofs are just outlined. The details are provided in the PhD Thesis [**15**] of the second author and will be also given elsewhere [**6**].

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2. Formulation of the problem

Let M be a finite graph smoothly imbedded into $\mathbb{R}^2$ (as it will be explained in the last section, all the results have natural analogs for the case of periodic infinite graphs and corresponding periodic thin domains). It is assumed that each edge is a finite C^2–curve and that edges intersect transversally at the graph's vertices. In some parts of the paper we will assume that edges are straight (although this is just a simplifying assumption and analogous results are most surely true in a much more general situation). Graph M has finite sets of vertices $V = \{v_l, l = 1, 2, ..., d\}$ and edges $\{M_j\}$. The "fattened graph" domain M^ε is the union of narrow strips ("pipes") M_j^ε of width of order ε surrounding edges M_j and of small neighborhoods U_l^ε of radius of order ε^α of vertices v_l, where $1 \geq \alpha > 0$. Here ε is a small parameter. The detailed structure of the domain will be specified in the sections below (see also Fig. 1).

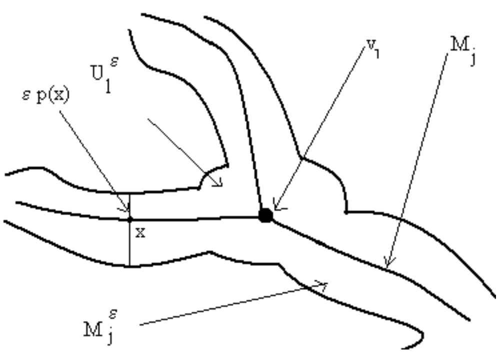

FIGURE 1. Local structure of M^ε.

On each strip M_j^ε we establish a local coordinate system (x_j, y_j) (or just (x, y) when there is no possibility of confusion), where x_j is the arc length coordinate along the edge M_j and y_j is measured along the normal directions to M_j. One can naturally define the space $L^2(M)$ consisting of functions defined along edges and square integrable along each edge. Analogously, one can define the Sobolev space $H^1(M)$ that consists of all functions that belong to H^1 on each edge and are contiuous at each vertex. This allows one to define second order differential operators on M, which will be extensively used in the following subsections.

The Neumann Laplace operator $-\triangle_{\varepsilon}$ in $L^2(M^{\varepsilon})$ is defined in the standard way by means of its quadratic form

$$e^{\varepsilon}[u,u] = \int\limits_{M} |\nabla u|^2 \, dA$$

with the domain $H^1(M^{\varepsilon})$. Here dA denotes the area element.

Our goal is to understand the behavior of the spectrum $\sigma(-\Delta_{\varepsilon})$ of this operator when $\varepsilon \to 0$. The considerations of [**1, 2**] suggest that this behavior probably depends on the value of α. Namely, one expects to see three different cases: when $0 < \alpha < \frac{1}{2}$, $\alpha = \frac{1}{2}$, and $\alpha > \frac{1}{2}$. These correspond to the situations when the area of the vertex neighborhouds dominates the area of the pipes, or both have the same order, or the area of the pipes dominates. The case when $\alpha > \frac{1}{2}$ was studied in [**5, 11**], and we describe the results of these papers below. The main aim of this paper is to address the two other cases.

3. Statement of the results

In this section we will present the results and outline their proofs. As it has already been mentioned, we will have to deal separately with three distinct cases: $\frac{1}{2} < \alpha \leq 1$, $0 < \alpha < \frac{1}{2}$, and the borderline case $\alpha = \frac{1}{2}$. In all these cases we will present a spectral problem in the space L^2 on the graph or in some its extension, such that the eigenvalues of the Neumann Laplacian $-\Delta_{\varepsilon}$ on the thin domain converge, when ε tends to zero, to the corresponding eigenvalues of the problem on the graph. The proofs in all cases follow the same technique. Namely, let us want to show convergence of the eigenvalues of $-\Delta_{\varepsilon}$ to the eigenvalues of an operator A acting on a functional space on the graph and having the quadratic form $e[u,u]$. In view of the min–max representation of the eigenvalues, it is sufficient to construct some "extension" and "averaging" operators Q_{ε} and P_{ε} such that Q_{ε} extends functions from the graph to the thin domain and P_{ε} acts in the opposite direction, and such that both operators for small values of ε almost do not increase the Rayleigh ratios of A and of $-\Delta_{\varepsilon}$ respectively, i.e.

$$\frac{e^{\varepsilon}[Q_{\varepsilon}u, Q_{\varepsilon}u]}{(Q_{\varepsilon}u, Q_{\varepsilon}u)} \leq (1+O(1))\frac{e[u,u]}{(u,u)} + O(1)$$

and

$$\frac{e[P_{\varepsilon}u, P_{\varepsilon}u]}{(P_{\varepsilon}u, P_{\varepsilon}u)} \leq (1+O(1))\frac{e^{\varepsilon}[u,u]}{(u,u)} + O(1).$$

Since the min-max representation of the eigenvalues requires control over the dimensions of subspaces, one also needs to guarantee that these operators when ε is small have only the trivial kernel on the eigenvalue subspaces of the corresponding problems. This usually follows immediately from the estimates provided in the lemmas below.

3.1. Small protrusions at the vertices: $\frac{1}{2} < \alpha \leq 1$. We summarize here some results of [**8**]–[**11**] and [**5, 6, 14, 15**]. Assume that the tube along the edge M_j has width $\varepsilon\, p_j(x)$, where $p_j(x) > 0$ is a C^1 function on the edge and x is the arc length coordinate. Notice that the width function p can be discontinuous at the vertices. Each vertex neighborhood is assumed to be contained in a ball of radius of order $\sim \varepsilon^\alpha$ and starshaped with respect to a smaller interior ball of a radius of the same order of smallness.

Consider the Schrödinger operator H_ε in M^ε

$$H_\varepsilon(\mathbf{A}, q) = (\frac{1}{i}\nabla - \mathbf{A}(x))^2 + q(x),$$

where the scalar electric $q(x)$ and vector magnetic $\mathbf{A}(x)$ potentials are defined in a fixed neighborhood of M, q is of the Lipschitz class, and $\mathbf{A}$ belongs to C^1. We impose Neumann conditions on ∂M^ε.

Let us also introduce the following operator $H(\mathbf{A}, q)$ on M:

$$H(\mathbf{A}, q)f(x_j) = -\frac{1}{p}(\frac{d}{dx_j} - iA_j^\tau(x))p(\frac{d}{dx_j} - iA_j^\tau(x))f + qf,$$

where we use $q(x)$ to denote the restriction of the potential q to M and A_j^τ is the tangential component of the field $\mathbf{A}$ to the edge M_j of M. The complete description of the operator requires us to impose some boundary conditions at vertices. These are:

1. f is continuous through each vertex.
2. at each vertex v

$$\sum_{\{j\,|\,v\in M_j\}} p_j\,(v)\left(\frac{df_j}{dx_j} - iA_j^\tau f_j\right)(v) = 0.$$

Here p_j denotes the function that provides the width of the tube along M_j (see the description of the domain above). The values $p_j\,(v)$ at the same vertex can be different for different edges M_j adjacent to v.

The next theorem summarizes some of the results of [**8**]–[**11**] and [**5, 6, 14, 15**]:

THEOREM 3.1. *For any $n = 1, 2, \ldots$*

$$\lim_{\varepsilon\to 0} \lambda_n(H_\varepsilon(\mathbf{A}, q)) = \lambda_n(H(\mathbf{A}, q)),$$

where λ_n is the n-th eigenvalue counted in increasing order (taking into account multiplicities).

This theorem shows that the asymptotic behavior of the spectrum of $H_\varepsilon(\mathbf{A},q)$ when $\varepsilon\to 0$ is given by the spectrum of the graph operator $H(\mathbf{A},q)$. The situation changes, however, when we allow the vertex neighborhoods to decay slower, as the results of the following subsections show.

3.2. Large protrusions at vertices: $0<\alpha<\frac{1}{2}$. In this and the next subsection we will assume that all edges are segments of straight lines and the strips are symmetric about the corresponding edges and have width 2ε. The vertex neighborhoods U_l^ε are assumed to be disks in $\mathbb{R}^2$ centered at v_l with radii ε^α and appropriately flattened where they join the tubes (see Fig. 2). We will also restrict our considerations to the Neumann Laplace operator only, albeit without any doubt similar results should hold for more general Schrödinger operators of the type considered in the previous section.

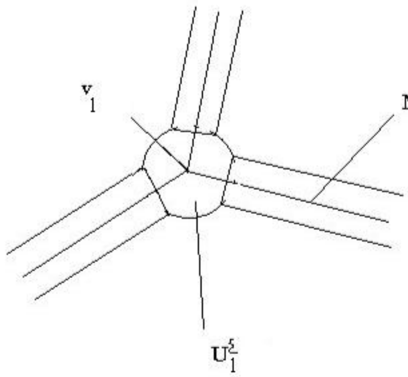

FIGURE 2. Local structure of M^ε in the case of large protrusions.

We say that $u\in H^1(M)$ belongs to $H_0^1(M)$ if u vanishes at all graph vertices. We denote by $\langle u,a\rangle$ elements of the Hilbert space $L^2(M)\oplus\mathbb{C}^d$ (or $H_0^1(M)\oplus\mathbb{C}^d$). Here d, as before, denotes the number of vertices, and $a=(a_1,...,a_d)$. Let us define a quadratic form on the Hilbert space $L^2(M)\oplus\mathbb{C}^d$ by

$$e\left[\langle u,a\rangle,\langle u,a\rangle\right]=\sum_j\int\limits_{M_j}\left|u_j'\right|^2dx_j \tag{3.1}$$

with the domain $H_0^1 \oplus \mathbb{C}^d$. This form is closed and positive and hence defines a positive self-adjoint operator A.

The domain $\mathcal{D}(A)$ can be easily described. In particular, for any $\langle u, a\rangle \in \mathcal{D}(A)$ we have $u \in H^2(M_j)$ for each edge M_j and u satisfies homogeneous Dirichlet boundary conditions at all vertices. The operator acts on such elements as

$$A\left\langle f\left(x_j\right), a\right\rangle = \left\langle -\frac{d^2 f}{dx_j^2}, 0\right\rangle \ , \langle f, a\rangle \in \mathcal{D}\left(A\right) \ , x_j \in M_j.$$

In other words, the space $\mathbb{C}^d$ of the "vertex states" belongs to $Ker\, A$, while on each edge the operator acts as the second derivative with Dirichlet boundary conditions. In particular, the spectrum of A is immediately computable from the knowledge of the length of all edges. Namely, let l_j be the length of the edge M_j. Then the spectrum $\sigma(A)$ of operator A consists of zero with multiplicity d (the number of vertices) and of the collection of sequences $(\pi n/l_j)^2$, $j = 1, 2, ...$ So, as far as the operator A is concerned, all vertices and edges are decoupled.

The spectra of both operators $-\Delta_\varepsilon$ and A are discrete. We denote the eigenvalues counted in increasing order with their multiplicity by $\lambda_n\left(-\Delta_\varepsilon\right)$ and $\lambda_n\left(A\right)$ respectively.

THEOREM 3.2. *Let* $0 < \alpha < \frac{1}{2}$. *Then*

$$\lim_{\varepsilon\to 0} \lambda_n\left(-\triangle_\varepsilon\right) = \lambda_n\left(A\right)$$

for each $n = 1, 2, ...$

The idea of the proof, as it was explained before, is based on the variational principle and has been emloyed in similar situations before [**11, 14, 12, 5**]. Namely, let T be one of the operators that we consider. Then we can use the standard min-max formula for the eigenvalues:

$$\lambda_n = \inf_{\dim W = n} \sup_{u(\neq 0)\in W} R\left(u\right)$$

Here

$$R\left(u\right) = \frac{\left(Tu, u\right)}{\left(u, u\right)}$$

is the Rayleigh quotient, λ_n is the n-th eigenvalue (counted in non-decreasing order) of the self-adjoint operator T, and W is an n-dimensional subspace of the domain of the quadratic form. Now one needs to be able to transplant functions from $H^1(M^\varepsilon)$ to $H_0^1(M) \oplus \mathbb{C}^d$ and back in such a way that the Rayleigh ratio does not increase much. Let us introduce now the necessary "extension" $Q^\varepsilon : H_0^1(M) \oplus \mathbb{C}^d \to H^1\left(M^\varepsilon\right)$ and "averaging" $P^\varepsilon : H^1\left(M^\varepsilon\right) \to H_0^1(M) \oplus \mathbb{C}^d$ mappings.

Let $\beta \in (\alpha, 2\alpha)$. Suppose that vertex v_l is an endpoint of edge M_j and its local coordinate along M_j is a_j. Let B_l^ε and D_l^ε be the disks centered at v_l with radii ε^β and $\varepsilon^\beta/2$ respectively. When ε is small enough, $D_l^\varepsilon \subseteq B_l^\varepsilon \subseteq U_l^\varepsilon$. We introduce the points

$$a_{j,\varepsilon}^1 = a_j + \varepsilon^\beta \sqrt{1 - r_l^2}$$

and

$$a_{j,\varepsilon}^2 = a_j + \varepsilon^\alpha \sqrt{1 - \varepsilon^{2-2\alpha}},$$

where r_l are constants such that the segments of lines $x_j = a_{j,\varepsilon}^1$ that belong to B_l^ε are disjoint for different values of j. Points $b_{j,\varepsilon}^1$, $b_{j,\varepsilon}^2$ are analogously intorduced at the other end of the edge M_j.

Let h be the piecewise linear function on $[a_{j,\varepsilon}^1, b_{j,\varepsilon}^1]$ that is defined as

$$h(x_j) = r_l \varepsilon^\beta + \frac{r_l \varepsilon^\beta - \varepsilon}{a_{2,j} - a_{1,j}} (x_j - a_{1,j})$$

on $[a_{j,\varepsilon}^1, a_{j,\varepsilon}^2]$, defined analogously on $[b_{j,\varepsilon}^2, b_{j,\varepsilon}^1]$, and is identically equal to ε on $[a_{j,\varepsilon}^2, b_{j,\varepsilon}^2]$.

Then we define for $x_j \in \left[a_{j,\varepsilon}^1, b_{j,\varepsilon}^1\right]$ the normal average of a function u on M_j^ε as

$$N_j u(x_j) = \frac{1}{2h(x)} \int_{-h(x_j)}^{h(x_j)} u(x_j, y_j)\, dy_j.$$

Let B be the unit disk in $\mathbb{R}^2$ centered at the origin. Consider a function $\omega \in C_0^\infty(B)$ of unit average. One can translate and homothetically shrink ω to produce functions $\omega_{l,\varepsilon} \in C_0^\infty(D_l^\varepsilon)$ of unit average. We define now the weighted average

$$c(v_l) = \frac{1}{\text{area of } D_l^\varepsilon} \int_{D_l^\varepsilon} \omega_{l,\varepsilon} u$$

We denote by $\psi_j : [a_{j,\varepsilon}^2, b_{j,\varepsilon}^2] \to [a_j, b_j]$ the 1-to-1 linear mapping between these two segments. A cut-off function ρ will also be used, such that

$$\rho(x) \in C_0^\infty(\mathbf{R}),\ \rho(0) = 1,\ \rho(x) = 0 \text{ when } |x| > 0.5 \min_j \left|b_{j,\varepsilon}^1 - a_{j,\varepsilon}^1\right|.$$

We are ready to define the "averaging" and "extension"mappings P^ε and Q^ε:

DEFINITION 3.3. Given $u \in H^1(M^\varepsilon)$, we define

$$P^\varepsilon u = \langle Pu, a \rangle,$$

where

$$Pu = \begin{cases} N_j u - N_j u\left(a^1_{j,\varepsilon}\right)\rho\left(x - a^1_{j,\varepsilon}\right) \\ -N_j u\left(b^1_{j,\varepsilon}\right)\rho\left(x - b^1_{j,\varepsilon}\right), & \text{if } x \in \left[a^1_{j,\varepsilon}, b^1_{j,\varepsilon}\right] \\ 0, & \text{if } x \in \left[a_j, a^1_{j,\varepsilon}\right] \cup \left[b^1_{j,\varepsilon}, b_j\right] \end{cases}$$

and $a = (a_1, a_2, ...a_d)$, where each a_l is defined as

$$a_l = \frac{c\,(v_l)\sqrt{\text{area of } U^\varepsilon_l}}{\sqrt{2\varepsilon}}.$$

DEFINITION 3.4. Let $\langle u, a\rangle \in H^1_0(M) \oplus \mathbb{C}^d$ and M_j be an edge of the graph M with the endpoints v_l and v_k. We define a function $Q^\varepsilon\langle u, a\rangle$ on M^ε as follows:

if $(x, y) \in U^\varepsilon_l$,

$$Q^\varepsilon\langle u, a\rangle = \frac{\varepsilon^{1/2-\alpha}a_l}{c_l}, \tag{3.2}$$

if $(x, y) \in U^\varepsilon(v_k)$,

$$Q^\varepsilon\langle u, a\rangle = \frac{\varepsilon^{1/2-\alpha}a_k}{c_k}, \tag{3.3}$$

and if $(x, y) \in M_j^{\ \varepsilon}$,

$$Q^\varepsilon\langle u, a\rangle = \frac{\varepsilon^{1/2-\alpha}a_l}{c_l}\rho(x_j - a^1_{j,\varepsilon}) + \frac{\varepsilon^{1/2-\alpha}a_k}{c_k}\rho(x_j - b^1_{j,\varepsilon}) + u \circ \psi_j. \tag{3.4}$$

Here x_j is, as before, the arc length coordinate along the edge M_j, and

$$c_l = \frac{\sqrt{\text{Area of } U^\varepsilon_l}}{\sqrt{2}\varepsilon^{\,\alpha}}.$$

Formulas (3.2) and (3.3) mean that in a vicinity of any vertex v_l the value of $Q^\varepsilon\langle u, a\rangle$ is defined as a constant determined by the l-th component of a. The third term in (3.4) is obtained as follows: we shrink u on each edge to a little bit shorter segment, then we extend it to a function which is defined on M^ε_j and independent on the normal variable. However, this extension would not fit continuously with the constant values (3.2) on the common boundary of U^ε_l and M^ε_j. In order to correct this, the two additional terms are added in (3.4) to adjust the values of $Q^\varepsilon\langle u, a\rangle$ close to the vertices.

For these two operators, the following properties hold (see [**6, 15**]):

LEMMA 3.5. *1. For any sufficiently small $\varepsilon > 0$, the operator Q^ε is continuous from $H^1_0(M) \oplus \mathbb{C}^d$ to $H^1(M^\varepsilon)$ and P^ε is continuous from $H^1(M^\varepsilon)$ to $H^1_0(M) \oplus \mathbb{C}^d$.*

2. For any sufficiently small $\varepsilon > 0$ *and for any* $\langle u, a\rangle \in H_0^1(M) \oplus \mathbb{C}^d$*, the following inequalities hold:*

$$\|Q^\varepsilon \langle u, a\rangle\|^2_{L^2(M^\varepsilon)} \geq 2\varepsilon\,(1 + O(\varepsilon^\gamma))\,\|\langle u, a\rangle\|^2_{L^2(M)\oplus\mathbb{C}^d}\,, \tag{3.5}$$

and

$$e^\varepsilon\left[Q^\varepsilon \langle u, a\rangle, Q^\varepsilon \langle u, a\rangle\right] \leq 2\varepsilon\,(1 + O(\varepsilon^\gamma))\left[e\left[\langle u, a\rangle, \langle u, a\rangle\right] + O(\varepsilon^\gamma)\,\|\langle u, a\rangle\|^2_{L^2(M)\oplus\mathbb{C}^d}\right], \tag{3.6}$$

where $\gamma > 0$ *is a constant independent of* u *and* ε*.*

3. For any sufficiently small $\varepsilon > 0$ *and for all* $u \in H^1(M^\varepsilon)$

$$e\left[P^\varepsilon u, P^\varepsilon u\right] \leq \frac{1}{2\varepsilon}\left[(1 + O(\varepsilon^\gamma))\,\|\nabla u\|^2_{L^2(M^\varepsilon)} + O(\varepsilon^\gamma)\,\|u\|^2_{L^2(M^\varepsilon)}\right] \tag{3.7}$$

and

$$\|P^\varepsilon u\|^2_{L^2(M)\oplus\mathbb{C}^d} \geq \frac{1}{2\varepsilon}\left[(1 + O(\varepsilon^\gamma))\,\|u\|^2_{L^2(M^\varepsilon)} + O(\varepsilon^\gamma)\,\|\nabla u\|^2_{L^2(M^\varepsilon)}\right]. \tag{3.8}$$

where $\gamma > 0$ *is a constant independent of* u *and* ε*.*

The lemma together with the min-max principle for eigenvalues leads to the following Corollary.

COROLLARY 3.6. There is a constant $\gamma > 0$ such that the following inequalities hold for any $n = 1, 2, 3, ...$:

$$\lambda_n(-\Delta_\varepsilon) \leq (1 + O(\varepsilon^\gamma))\,\lambda_n(A) + O(\varepsilon^\gamma) \quad \text{when } \varepsilon \to 0 \tag{3.9}$$

$$\lambda_n(A) \leq \lambda_n(-\Delta_\varepsilon) + O(\varepsilon^\gamma) \tag{3.10}$$

Theorem 3.2 now follows from Corollary 3.6.

3.3. The borderline case: $\alpha = \frac{1}{2}$**.** This time, we define the following positive closed quadratic form on $L^2(M) \oplus \mathbb{C}^d$:

$$\widetilde{e}\left[\langle u, a\rangle, \langle u, a\rangle\right] = \sum_j \int\limits_M \left|u_j'\right|^2 dx_j \tag{3.11}$$

with the domain $\mathcal{D} \subseteq H^1 \oplus \mathbb{C}^d$ consisting of elements $\langle u, a\rangle \in H^1 \oplus \mathbb{C}^d$ such that $a_l = \sqrt{\frac{\pi}{2}}u(v_l)$ for each $l = 1, ..., d$. This form defines a positive self-adjoint operator $\widetilde{A}$. One can show by the direct calculations that the operator acts on elements $\langle f, a\rangle \in \mathcal{D}\left(\widetilde{A}\right)$ as

$$\widetilde{A}\langle f, a\rangle = \left\langle -\frac{d^2 f}{dx_j^2}, c\right\rangle, \; x_j \in M_j.$$

where $c = (c_1, ..., c_d) \in \mathbb{C}^d$ is defined by

$$c_k = \sqrt{\frac{2}{\pi}} \sum_{v_k \in M_l} f_l' (v_k), k = 1, 2, ..., d.$$

The eigenvalue problem $\widetilde{A} \langle f, a \rangle = \lambda \langle f, a \rangle$ can then be rewritten as

$$\begin{cases} -\frac{d^2 f}{dx_j^2} = \lambda f & \text{when} & x \in M_j \\ f \text{ is continous} & \text{at each vertex } v_k & k = 1, ..., d \\ \sum\limits_{v_k \in M_l} f_l' (v_k) = \frac{\lambda \pi}{2} f (v_k) & \text{at each vertex } v_k & k = 1, ..., d \end{cases}.$$

In this formulation of the spectral problem the extra variables a are eliminated. So the spectral problem is rewritten on the graph alone, at the expense of the spectral parameter now appearing also in the boundary conditions.

As in the previous cases, the spectra of both positive operators $-\Delta_\varepsilon$ and $\widetilde{A}$ are discrete.

THEOREM 3.7. *The equality*

$$\lim_{\varepsilon \to 0} \lambda_n (-\Delta_\varepsilon) = \lambda_n \left(\widetilde{A} \right)$$

holds for each $n = 1, 2, ...$

The idea of the proof is similar to the one employed in the proofs of the results of the two previous subsections. However, the construction of the "extension" and "averaging" operators needs modification.

We will use the same linear function ψ_j and cut-off function $\rho(x)$ as in previous section.

Let $\beta \in (0.5, 1)$ and v_l be a vertex with the local coordinate a_j along the edge M_j. Considers disks B_l^ε and D_l^ε centered at v_l with radii ε^β and $\varepsilon^\beta / 2$ correspondingly. For small values of ε, we have $D_l^\varepsilon \subseteq B_l^\varepsilon \subseteq U_l^\varepsilon$. Let

$$a_{j,\varepsilon}^1 = a_j + \varepsilon^\beta \sqrt{1 - r_l^2}$$

and

$$a_{j,\varepsilon}^2 = a_j + \sqrt{\varepsilon - \varepsilon^2},$$

where the numbers r_l are chosen as in the previous subsection. The points $b_j, b_{j,\varepsilon}^1$, and $b_{j,\varepsilon}^2$ are defined analogously at the other end of the edge M_j. With this choice of points $a_{j,\varepsilon}^1$ and $a_{j,\varepsilon}^2$, we define the function h as before: it is equal to ε on $[a_{j,\varepsilon}^2, b_{j,\varepsilon}^2]$ and as

$$h(x_j) = r_l \varepsilon^\beta + \frac{r_l \varepsilon^\beta - \varepsilon}{a_{j,\varepsilon}^1 - a_{j,\varepsilon}^2} \left(x_j - a_{j,\varepsilon}^1 \right)$$

on $[a^1_{j,\varepsilon}, a^2_{j,\varepsilon}]$ and analogously on $[b^2_{j,\varepsilon}, b^1_{j,\varepsilon}]$. Then for $x_j \in [a^1_{j,\varepsilon}, b^1_{j,\varepsilon}]$ we define the normal average of u as

$$N_j u(x_j) = \frac{1}{2h(x_j)} \int_{-h(x_j)}^{h(x_j)} u(x_j, y_j)\, dy_j.$$

In the following definition, we use the same notations as before: an edge M_j has the endpoints v_l and v_k.

Definition 3.8. For any $\langle u, a\rangle \in \mathcal{D}$, define $Q^\varepsilon \langle u, a\rangle$ on M^ε as follows:

if $(x, y) \in U^\varepsilon_l$,

$$Q^\varepsilon \langle u, a\rangle = u(v_l); \tag{3.12}$$

if $(x, y) \in U^\varepsilon_k$,

$$Q^\varepsilon \langle u, a\rangle = u(v_k); \tag{3.13}$$

if $(x, y) \in M^\varepsilon_j$,

$$Q^\varepsilon \langle u, a\rangle (x, y) = u \circ \psi_j(x). \tag{3.14}$$

Definition 3.9. Given $u \in H^1(M^\varepsilon)$, define

$$P^\varepsilon u = \langle Pu, a\rangle,$$

where $a = (a_1, a_2, ..., a_d) \in \mathbb{C}^d$,

$$a_l = \sqrt{\frac{\pi}{2}} c(v_l),$$

and Pu is defined as follows:

if $x \in [a_j, a^1_{j,\varepsilon}]$,

$$Pu = c(v_l);$$

if $x \in [b^1_{j,\varepsilon}, b_j]$,

$$Pu = c(v_k);$$

if $x \in [a^1_{j,\varepsilon}, b^1_{j,\varepsilon}]$,

$$\begin{aligned} Pu = N_j u &+ \left(c(v_l) - N_j u\left(a^1_{j,\varepsilon}\right)\right) \rho\left(x - a^1_{j,\varepsilon}\right) \\ &+ \left(c(v_k) - N_j u\left(b^1_{j,\varepsilon}\right)\right) \rho\left(x - b^1_{j,\varepsilon}\right). \end{aligned}$$

The crucial step is now the following lemma:

Lemma 3.10. *1. For any sufficiently small $\varepsilon > 0$ the operator Q^ε maps $\mathcal{D}$ into $H^1(M^\varepsilon)$ and the operator P^ε maps $H^1(M^\varepsilon)$ into $\mathcal{D}$*

2. For any sufficiently small $\varepsilon > 0$ and for any $\langle u, a\rangle \in \mathcal{D}$, the following inequalities hold:

$$\|Q^\varepsilon \langle u, a\rangle\|^2_{L^2(M^\varepsilon)} \geq 2\varepsilon (1 + O(\varepsilon)) \|\langle u, a\rangle\|^2_{L^2(M)\oplus\mathbb{C}^d}$$

$$e^{\varepsilon}\left[Q^{\varepsilon}\langle u,a\rangle, Q^{\varepsilon}\langle u,a\rangle\right] \leq 2\varepsilon\left(1+O\left(\varepsilon\right)\right) e\left[\langle u,a\rangle, \langle u,a\rangle\right]$$

where $\gamma > 0$ *is a constant independent of* u *and* ε.

3. For any sufficiently small $\varepsilon > 0$ *and for all* $u \in H^1\left(M^{\varepsilon}\right)$

$$e\left[P^{\varepsilon}u, P^{\varepsilon}u\right] \leq \frac{1}{2\varepsilon}\left(1+O\left(\varepsilon^{\gamma}\right)\right)\|\nabla u\|^2_{L^2(M^{\varepsilon})}$$

and

$$\|P^{\varepsilon}u\|^2_{L^2(M)\oplus\mathbb{C}^d} \geq \frac{1}{2\varepsilon}\left[\left(1+O\left(\varepsilon^{\gamma}\right)\right)\|u\|^2_{L^2(M^{\varepsilon})} + O\left(\varepsilon^{\gamma}\right)\|\nabla u\|^2_{L^2(M^{\varepsilon})}\right].$$

where $\gamma > 0$ *is a constant independent of* u *and* ε.

This lemma together with the min-max representation of eigenvalues leads to

Corollary 3.11. The following inequalities hold for any $n = 1, 2, 3, ..$:

$$\lambda_n\left(-\triangle_{\varepsilon}\right) \leq \left(1+O\left(\varepsilon\right)\right)\lambda_n\left(A\right), \quad \text{when } \varepsilon \to 0.$$

$$\lambda_n\left(A\right) \leq \lambda_n\left(-\triangle_{\varepsilon}\right) + O\left(\varepsilon^{\gamma}\right)$$

where $\gamma > 0$ is a constant.

Theorem 3.7 is now an immediate consequence of Corollary 3.11.

4. Concluding remarks

- The decoupling of edges and vertices occuring when $\alpha < 0.5$ can be understood easily. Namely, for small values of ε the tubes are extremely narrow in comparison with the vertex neighborhoods. This means that a particle entering along a tube one of the vertex vicinities has a negligible chance to get out. So, the vertex vicinity works as a "black hole" for particles, which means that in the limit Dirichlet boundary conditions are to be imposed. On the other hand, among the eigenstates sitting inside the "black hole" only the ground state survives in the limit.
- As it has already been mentioned, one can consider graphs that are periodic with respect to a lattice rather than finite. The thin domains M^{ε} are also considered to be periodic. Then one is dealing with absolutely continuous rather than discrete spectra. In this case one can use Floquet theory and impose cyclic Floquet/Bloch conditions corresponding to a quasimomentum (see the basic notions of the Floquet theory e.g., in [**3, 7**]). For any fixed quasimomentum one deals with a finite graph on the torus, while presence of a quasimomentum in

the operator can be interpreted as electric and magnetic potentials entering the picture. So, one can literally repeat the proofs of the results of this situation and obtain convergence of Bloch (Floquet) eigenvalues uniformly with respect to the quasimomentum. This, according to Floquet theory, implies convergence of spectra.

- It should be possible (and probably not hard) to carry over all the results of this paper to graphs with smooth non-straight edges, non-constant width tubes M_j^ε, and presence of electric and magnetic potentials.
- A certain kind of resolvent convergence of the Neumann Laplacian on a fattened graph domain to an operator on the graph was shown in [**12, 13**] in the particular case when the graph is a tree.
- Convergence of solutions of the heat equations in a fattened graph domain with Neumann conditions to solutions of the heat equation on the limit graph (in absence of potentials) was shown in [**1, 2**].

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