APPLICATION OF EXACT DIFFERENCE SCHEMES TO THE CONSTRUCTION AND STUDY OF DIFFERENCE SCHEMES FOR GENERALIZED SOLUTIONS

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R. D. LAZAROV, V. L. MAKAROV AND A. A. SAMARSKI

ABSTRACT. This paper discusses the application of exact difference schemes to the construction and study of difference schemes for generalized solutions.

Bibliography: 15 titles.

§1. Introduction

In the case of boundary value problems for ordinary second order differential equations with discontinuous coefficients, Tihonov and Samarskii (see [1] and [2]) have studied in detail the convergence rate of homogeneous difference schemes and constructed exact schemes and schemes of any order of accuracy for nonsmooth solutions. However, their methods, which permitted them to obtain very sharp a priori estimates and convergence theorems, do not carry over to the multidimensional case.

The traditional approach to the study of the convergence rate of difference schemes by the method of energy inequalities has two weak points: first, the error of the approximation is analyzed by means of Taylor’s classical formula, which requires too high a degree of smoothness of the desired solution; second, there is no procedure for obtaining unimprovable error estimates in weaker norms for nonsmooth solutions. From the point of view of the theory and application of difference methods, we are greatly interested in those estimates of the convergence rate in which the order is compatible with the smoothness of the solution of the differential problem in question.

We say that an estimate of the rate of convergence of a difference scheme is compatible with the smoothness of the required solution if it has the form

$$\| y - u \|_{W^s_2(\omega)} \leq M |h|^{k-s} \| u \|_{W^s_2(\Omega)}, \quad k > s,$$

where $k$ and $s$ are nonnegative integers and $\| \cdot \|_{W^s_2(\omega)}$ and $\| \cdot \|_{W^s_2(\Omega)}$ are the Sobolev norms on the set of functions of a discrete and continuous argument, respectively.

The above definition seems appropriate, since a) the formulation of a differential problem in the classes $W^k_2(\Omega)$ is natural, and b) estimates of this type are characteristic of the finite element method (see [3] and [4]); in this case the left-hand side of (1) contains

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the $W^2_0(\Omega)$-norm in the space of continuous functions, $k - 1$ is the degree of polynomial interpolations of discrete functions, and $s$ does not exceed half the order of the given differential equation. In certain cases these estimates are unimprovable. In this paper we obtain compatible a priori estimates of type (1) and an estimate for the convergence rate of difference schemes for generalized solutions of elliptic and parabolic equations. To this end we use the resolving operator of the exact difference schemes (see [1], [2], and [5]) and a lemma of Bramble and Gilbert [4].

Estimates of form (1) with $s = 1$ and $k = 2, 3$ have been obtained in [6] and [7] for a difference scheme for the first boundary value problem for Poisson’s equation and the Lamé equations of elasticity theory. Such estimates, with the loss of $1/2$ in the order of accuracy (that is, estimates of the order $O(h)^{k+s-0.5}$) have been obtained in [8] for difference schemes for fourth order equations with $s = 2$ and $k = 3, 4$ and in [9] for eigenvalues of a second order equation with $k = 2$.

In all these papers the convergence is investigated in the natural energy norms of the difference operator by means of classical a priori estimates for the difference solutions (see [10]). In studying the convergence of a difference scheme in norms that are weaker than the energy ones, a decisive role is played by the resolving operators of the exact difference schemes. Here essential use is made of the mathematical apparatus developed in [5].

The convergence in the $L_2(\omega)$-norm of a difference scheme for the equation $\Delta u = -f(x, u)$ in a rectangle has been considered in [11] and [12]. In [11] a nine-point difference scheme is constructed for $f(x, u) = -cu + f(x)$, $c = \text{const} > 0$, whose accuracy, depending on the desired smooth solution, is characterized by the estimate (1) with $s = 0$ and $k = 2, 3, 4$. The case considered in [12] is that when the right-hand side $f(x, u)$ satisfies the Lipschitz condition with respect to $u$. The estimate obtained there is of the form (1) with $s = 0$ and $k = 2$, provided the Lipschitz constant is sufficiently small.

§2. Auxiliary notation. Properties of the resolving operators of the exact difference schemes

Let $\Omega = \{x = (x_1, x_2): 0 < x_a < l_a, l_a > 0, a = 1, 2\}$ be a rectangle with boundary $\Gamma$. In $\Omega$ we introduce a nonuniform mesh $\omega = \omega_1 \times \omega_2$, where $\omega_a = \{x_a = x_{a,i}, i_a = 1, \ldots, N_a - 1\}, x_{a,i} - x_{a,i-1} = h_{a,i}, a = 1, 2, h = h_{a,i} = h_{a,i+1}$, and $h = 0.5(h_{a,i} + h_{a,i+1}).$

We denote by $\omega$ the uniform mesh defined by

$$\omega = \omega_1 \times \omega_2, \quad \omega_a = \{x_a = i_a h_a, i_a = 1, 2, \ldots, N_a - 1, h_a = l_a/N_a\}, \quad a = 1, 2.$$  

By $\gamma$ we note the mesh nodes lying on $\Gamma$, with the exception of the vertices of the rectangle. In what follows we use the standard notation for the norms in the Sobolev spaces $W^k_2(\Omega)$ of functions defined on $\Omega$ (see [13]). The analogs of the norms for functions given on $\omega$ are denoted by $W^k_2(\omega)$ (see [10]). Everywhere below $M$ denotes positive constants independent of the mesh step $h$, the solutions of our problem, and the difference scheme. The resolving operators $\hat{F}^x$ of the exact difference schemes on the nonuniform mesh are defined by (see [11] and [8])

$$\hat{F}^x(W(\cdot)) = \frac{\tilde{h}^{-1}h}{\alpha(0, h)} \int_{-1}^{0} \alpha(s, h)w^*(s)\, ds + \frac{\tilde{h}^{-1}h^+}{\beta(0, h^+)} \int_{0}^{1} \beta(s, h^+)w^*(s)\, ds,$$

$$w^*(s) = \begin{cases} w(x + sh), & -1 \leq s \leq 0, \\ w(x + sh^+), & 0 \leq s \leq 1, \end{cases}$$  

where $\alpha(s, h) = h\nu_1(x)$ and $\beta(s, h^+) = h^+\nu_2(x)$ are pattern functions connected with the
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The differential operator

\[ L^{(k,q)}u = \frac{d}{dx} \left[ k(x) \frac{du}{dx} \right] - q(x)u, \quad (3) \]

in which

\[ 0 < c_1 \leq k(x) \leq c_2, \quad k(x) \in L_1(0,1), \quad q(x) \geq 0, \quad q(x) \in L_p(0,1), \quad p \geq 1. \quad (3') \]

In the case of the uniform mesh we use the notation \( T^x \). In order to distinguish the variable with respect to which the resolving operator acts when it is defined on functions of two variables \( x_1 \) and \( x_2 \), we endow it with an additional index as follows:

\[ \hat{T}^x(u) = \hat{T}^{x_1}(u(\cdot, x_2)), \quad \hat{T}^{x_2}(u) = \hat{T}^{x_2}(u(x_1, \cdot)). \]

We denote by \( T^2_a \) the operator \( T^x \) for \( k = k_a(x_a) \equiv 1 \) and \( q = q_a(x_a) \equiv 0 \). Then, as can be easily seen, this operator is defined by

\[ T^2_a(u) = T^2(u(\cdot, x_2)) \]

\[ = \frac{1}{h^2} \int_{x_1-h}^{x_1} (\xi - x_1 + h_1)u(\xi, x_2) d\xi + \frac{1}{h^2} \int_{x_1}^{x_1+h_1} (x_1 + h_1 - \xi)u(\xi, x_2) d\xi \]

\( (T^2_1 \) is defined analogously) and coincides with the square of the averaging Steklov operator \( T_1 \) given by

\[ T_1(u) = T_1(u(\cdot, x_2)) = \frac{1}{h_1} \int_{x_1-0.5h_1}^{x_1+0.5h_1} u(\xi, x_2) d\xi. \]

For \( k = k_a(x_a) \) and \( q = q_a(x_a) \) we set

\[ L^{(k,q)}u = L_a u, \quad \alpha = 1, 2. \]

Suppose \( L_a \) satisfies \((3')\). Then the pattern functions \( \alpha_i(s, h_i) \) and \( \beta_i(s, h_i) \) are monotone in \( s \) (\( \alpha_i(s, h_i) \) is increasing and \( \beta_i(s, h_i) \) decreasing for \( s \in (-1, 1), i = 1, 2 \), and we have

\[ |\hat{T}^{x_1}(w)| \leq \bar{h}_1 \int_{x_1-h_1}^{x_1+h_1} |w(\xi, x_2)| d\xi, \quad |\hat{T}^{x_2}(w)| \leq \bar{h}_2 \int_{x_2-h_2}^{x_2+h_2} |w(x_1, \xi)| d\xi. \quad (4) \]

One of the basic properties of the resolving operators of the exact difference schemes is expressed by the equality

\[ \hat{T}^x(L_a u) = (a_a u(x_\alpha))_\alpha - d_\alpha u \equiv \Lambda_\alpha u, \quad x_\alpha \in \omega_\alpha, \alpha = 1, 2, \quad (5) \]

where \( a_\alpha \) and \( d_\alpha \) are defined by

\[ a_\alpha = a_\alpha(x) = [\alpha_i(0, h_i)]^{-1}, \quad d_\alpha = d_\alpha(x) = \hat{T}^{x_1}(q_i (\cdot)). \]

We give another formula for \( \hat{T}^x(1) \), which is needed below:

\[ \hat{T}^x(1) = \bar{h}_1 a(0, h) \int_{-1}^{0} a(s, h) ds + \bar{h}_2 \beta(0, h) \int_{1}^{0} \beta(s, h) ds. \]

§3. The convergence of difference schemes

for solutions in \( W_2^1(\Omega) \)

We consider the following elliptic problem, in which the differential operator is an operator with separable variables:

\[ Lu = L_1 u + L_2 u = -f(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \Gamma, \]

\[ L_a u = \frac{\partial}{\partial x_\alpha} \left( k_a(x_\alpha) \frac{\partial u}{\partial x_\alpha} \right) - q_a(x_\alpha) u, \quad \alpha = 1, 2. \quad (6) \]
We approximate problem (6) by the difference scheme
\[ Ay = \left\{ b_2 \Lambda_1 + b_1 \Lambda_2 \right\} y = -T^{x_2}T^{x_1}f = \varphi(x), \quad x \in \omega, \]
\[ y(x) = 0, \quad x \in \gamma, \quad b_{a} = T^{x_a}(1), \quad a = 1, 2, \]
where the resolving operator \( T^{x_a} \) of the exact difference scheme in the direction \( x_a \) is defined by (2) and the difference operators \( \Lambda_a \) by (5). We note that \( T^{x_1}T^{x_2} = T^{x_2}T^{x_1} \).

We show that if the solution of (6) is in \( W_2^1(\Omega) \), then an analog of (1) holds with \( s = 0 \) and \( k = 1 \). The special feature of this estimate is the fact that the solution \( u(x) \), as a function in \( W_2^1 \), may be undefined at the mesh nodes. Therefore, we equate the difference solution \( y(x) \) with some average \( \bar{u}(x) \) of the solution \( u(x) \) in the neighborhood of a node \( x \in \omega \). Of all the various possibilities we choose the simplest one. Thus, in what follows we assume that \( \bar{u}(x) \) is defined by
\[ \bar{u}(x) = \begin{cases} T_1T_2u(x), & x \in \omega, \\ 0, & x \in \gamma, \end{cases} \text{ or } \bar{u}x = \begin{cases} T_1^2T_2^2u(x), & x \in \omega, \\ 0, & x \in \gamma. \end{cases} \]

Below we estimate the convergence of the difference schemes (7) for the solutions of the given problem (6) in the class \( W_2^2(\Omega) \). We assume the coefficients of (6) satisfy the conditions
\[ 0 < c_1 \leq k_a \leq c_2, \quad k_a(x_a) \in L_1(\Omega), \quad q_a(x_a) \geq 0, \quad q_a \in L_p(\Omega), \quad p > 1, \quad a = 1, 2, \]
\[ f(x) = \sum_{\alpha=1}^{2} \frac{\partial f_a(x)}{\gamma x_\alpha} + f_0(x), \quad f_a(x) \in L_2(\Omega), \quad \alpha = 1, 2, \quad f_0(x) \in L_{2+e}(\Omega), \quad e > 0, \]

which ensure that \( u(x) \) belongs to \( W_2^1(\Omega) \) (see [13]). The following proposition holds.

**Theorem 1.** If conditions (8) are satisfied, then the difference scheme (7) has first order accuracy in \( L_2(\omega) \), and
\[ \|y - \bar{u}\|_{L_2(\omega)} \leq M |h| \|u\|_{W_2^1(\Omega)}, \]
where the constant \( M \) is independent of \( h \) and \( u(x) \).

We point out the main steps of the proof. For the function \( z = y - \bar{u} \) we obtain the difference problem
\[ \Lambda z = -\Psi(x), \quad x \in \omega, \quad z(x) = 0, \quad x \in \gamma, \]
where the error \( \Psi(x) \) of the approximation is represented in the form
\[ \Psi(x) = \sum_{\alpha=1}^{2} \Lambda_a \eta_a, \quad \eta_a = T^{x_2-a}(u) - b_{3-a} \bar{u}, \quad \alpha = 1, 2. \]

We easily see that the operators \(-b_{3-a}^{-1} \Lambda_a, \alpha = 1, 2, \) commute and are symmetric and positive in the sense of the inner product with weight \( b_1b_2 \). Developing the general operator approach set forth in [11], we obtain the estimates
\[ \left\| (b_1^{-1} \Lambda_1 + b_2^{-1} \Lambda_2)^{-1} b_{3-a} \Lambda_a \right\| \leq 1, \quad \alpha = 1, 2, \]
which lead to the inequality
\[ \|y b_1b_2 z\|_{L_2(\omega)} \leq \sum_{a=1}^{2} \|y b_1b_2 \eta_a\|_{L_2(\omega)}. \]
Hence, taking into account the properties of the pattern functions and the structure of the operators $T^{x*}$, we obtain

$$\| z \|_{L^2(\omega)} \leq M \sum_{\alpha=1}^{2} \| \eta_\alpha \|_{L^2(\omega)}. \quad (10)$$

The expressions $\eta_\alpha$, $\alpha = 1, 2$, are linear bounded functionals in $W^1_2(\Omega)$ which vanish for constants. Applying the Bramble-Gilbert lemma [4], we obtain the desired estimate (9).

**Remark 1.** (9) also remains valid for a difference scheme constructed by means of an arbitrary nonuniform mesh in the rectangle $\Omega$.

**Remark 2.** An analogous theorem also holds for quasilinear equations when only the right-hand side depends on $u(x)$.

**Remark 3.** Without significant modifications we can construct and study difference schemes for solutions in $W^1_2(\Omega)$ for boundary conditions of the second and third kind.

**Remark 4.** In the case of the Dirichlet problem for Poisson's equation in cylindrical coordinates with axial symmetry,

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} = -f(r, z), \quad (r, z) \in \Omega, \quad u(r, z) = 0, \quad (r, z) \in \Gamma,$$

an analog of Theorem 1 holds. The singularity in the coefficients of the equation creates additional difficulties in the proof of the theorem, but by carefully applying the technique developed above, we can obtain in this case a result analogous to the statement of Theorem 1 (see [14]).

**Remark 5.** For an elliptic equation with separable variables only in its principal part (which contains the derivatives)

$$(\tilde{L}_1 + \tilde{L}_2)u - q(x)u = -f(x), \quad \tilde{L}_\alpha = \frac{\partial}{\partial x_\alpha} \left( k_\alpha(x_\alpha) \frac{\partial}{\partial x_\alpha} \right), \quad \alpha = 1, 2,$$

$$(q(x) = q(x_1, x_2) \geq 0, \quad x \in \Omega,$$

we construct the scheme

$$\sum_{\alpha=1}^{2} T^{x_\alpha}(1)(\hat{\alpha}, y_{x_\alpha})_{x_\alpha} - \hat{T}^{x_1, x_2}(q)y = -\hat{T}^{x_1, x_2}(f), \quad x \in \omega, \quad y(x) = 0, \quad x \in \gamma,$$

where

$$\hat{\alpha} = h_\alpha \left( \int_{x_\alpha - h_\alpha}^{x_\alpha} \frac{d\xi}{k_\alpha(\xi)} \right)^{-1}$$

and the $\hat{T}^{x_\alpha}$ are the resolving operators of the exact difference schemes for $\hat{L}_\alpha = L_\alpha^{(k_\alpha, 0)}$, $\alpha = 1, 2$. Suppose the functions $k_\alpha(x_\alpha)$, $\alpha = 1, 2$, and $f(x)$ satisfy (8) and $q(x) \in L^2(\Omega)$. Then estimate (9) holds for this scheme. The proof is based on a special form of the error of the approximation and on the following proposition: Let $A_\alpha$ and $B$ be linear operators on a real Hilbert space $H$, and let

$$A_\alpha = A_\alpha^* \geq \nu_\alpha E, \quad \alpha = 1, 2, \quad A_1 A_2 = A_2 A_1, \quad B = B^* \geq 0, \quad \| B \| \geq \mu,$$

where $\nu_\alpha$ and $\mu$ are positive constants that do not depend on $h$. Then

$$\|(A_1 + A_2 + B)^{-1} A_\alpha\| \leq M, \quad \alpha = 1, 2,$$

where the constant $M$ depends only on $\nu_\alpha$ and $\mu$. 
§4. Coefficient stability

The practical application of the difference scheme (7) may give rise to specific difficulties connected with the determination of its coefficients. In order to construct a simpler difference scheme of the same order of accuracy, we need a lemma on the coefficient stability (co-stability) of (7). To obtain it, along with (7) we consider a "simpler" scheme of the form

\[ \bar{b}_2 \bar{\Lambda}_1 + \bar{b}_1 \bar{\Lambda}_2 \bar{y} = -\bar{\phi}(x), \quad x \in \omega, \quad \bar{y}(x) = 0, \quad x \in \gamma, \quad (11) \]

where \(\bar{\Lambda}_a y \equiv (\bar{a}_a y(x))_x - \bar{d}_a y\) and \(\bar{b}_a = \bar{b}_a(x), \alpha = 1, 2\). Then for the function \(v = y - \bar{y}\) we obtain the boundary value problem

\[ \Lambda v = -(\bar{\Lambda} - \Lambda)y - (\bar{\phi} - \phi) = -\bar{\Psi}, \quad x \in \omega, \quad v(x) = 0, \quad x \in \gamma, \quad (12) \]

where \(\Lambda\) is defined by (7).

We represent the first term in \(\bar{\Psi}\) in the form

\[ \Lambda y = (b_2 - b_1)\Lambda_1 + (b_1 - b_1)\Lambda_2 + b_2(\bar{\Lambda}_1 - \Lambda_1) + b_1(\bar{\Lambda}_2 - \Lambda_2) \]

We set

\[ \eta_0 = (b_2 - b_1)\Lambda_1 \bar{y} + (b_1 - b_1)\Lambda_2 \bar{y}, \quad \eta_\alpha = b_3 - \alpha(\bar{\Lambda}_1 - \Lambda_1) \bar{y}, \quad \alpha = 1, 2. \]

Let us now derive an a priori estimate for the function \(v\) in the mesh norm of \(W_2^1(\omega)\). To this end we consider the inner product with \(v\) of both sides of the equation in (12). Then

\[ ||\bar{y} - y||_{W_2^1(\omega)} \leq M \sum_{\beta = 0}^{2} |(\eta_\beta, \bar{y} - y)| + ||\phi - \bar{\phi}||_{W_2^{-1}(\omega)} ||\bar{y} - y||_{W_2^1(\omega)}. \quad (13) \]

Taking into account the inequalities

\[ |(\eta_0, v)| \leq M \max_\alpha ||\bar{b}_a - b_a||_{C(\omega)} ||\bar{\Psi}||_{W_2^1(\omega)} ||v||_{W_2^1(\omega)}, \]

\[ |(\eta_\alpha, v)| \leq M \max_\alpha ||\bar{b}_3 - \alpha||_{C(\omega)} \left[ ||\bar{a}_a - a_a||_{C(\omega)} + ||\bar{d}_a - d_a||_{W_2^{-1}(\omega)} \right] ||\bar{\Psi}||_{W_2^1(\omega)} ||v||_{W_2^1(\omega)}, \]

from (13) we obtain

\[ ||\bar{y} - y||_{W_2^1(\omega)} \leq M \left\{ \max_\alpha \left[ ||\bar{b}_a - b_a||_{C(\omega)} + ||\bar{a}_a - a_a||_{C(\omega)} \right. \right. \]

\[ \left. + ||\bar{d}_a - d_a||_{W_2^{-1}(\omega)} \right] + \left. ||\bar{b}_3 - a_a||_{W_2^{-1}(\omega)} \right) ||\bar{\Psi}||_{W_2^1(\omega)}. \]

This yields

**Lemma 1.** Suppose the coefficients \(b_a\) of the difference scheme (7) are bounded and that problem (11) has a unique solution that is bounded in \(W_2^1(\omega)\). Then

\[ ||\bar{y} - y||_{W_2^1(\omega)} \leq M \left\{ \max_\alpha \left[ ||\bar{b}_a - b_a||_{C(\omega)} + ||\bar{a}_a - a_a||_{C(\omega)} \right. \right. \]

\[ \left. + ||\bar{d}_a - d_a||_{W_2^{-1}(\omega)} \right] + \left. ||\bar{b}_3 - a_a||_{W_2^{-1}(\omega)} \right) ||\bar{\Psi}||_{W_2^1(\omega)}. \]

**Corollary 1.** Any difference scheme of the form (11) whose coefficients \(\bar{b}_a, \bar{a}_a\) and \(\bar{d}_a\), \(\bar{\phi}\) differ by \(O(1)\) in the norms of \(C(\omega)\) and \(W_2^{-1}(\omega)\), respectively, from the corresponding
coefficients and right-hand side of the scheme (7) has first order accuracy in $L_2(\omega)$. In particular, this holds for a scheme constructed by means of truncated one-dimensional schemes of rank zero.

§5. The construction of a second order scheme

for solutions in $W^2_2$

It is interesting to construct a difference scheme of second order accuracy when the solution of the given problem belongs to $W^2_2(\Omega)$.

We consider the problem with separable variables

$$Lu \equiv (L_1 + L_2)u = -f(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \Gamma,$$

where

$$L_\alpha u = \frac{\partial}{\partial x_\alpha} \left( k_\alpha(x_\beta - a) \frac{\partial u}{\partial x_\alpha} \right), \quad \alpha = 1, 2.$$

We assume

$$0 < c_1 \leq k_\alpha(x_\beta - a) \leq c_2, \quad k_\alpha \in L_1(\Omega), \quad \alpha = 1, 2, \quad f(x) \in L_2(\Omega).$$

These conditions ensure that the solution of (14) is of class $W^2_2(\Omega)$ (see [13]).

We construct a difference scheme of the form

$$\Lambda y = \sum_{\alpha=1}^{2} b_{3-\alpha} \Lambda_\alpha y = \sum_{\alpha=1}^{2} \hat{T}^{x_3 - x}(k_\alpha) y_{x_\alpha} y_{x_\alpha} = -\hat{T}^{x_1} \hat{T}^{x_2}(f), \quad x \in \tilde{\omega},$$

$$y(x) = 0, \quad x \in \gamma.$$

For the error $z = y - u$ we obtain the problem

$$\Lambda z = \sum_{\alpha=1}^{2} \eta_\alpha x_{a_\alpha} z_{a_\alpha}, \quad x \in \tilde{\omega}, \quad z(x) = 0, \quad x \in \gamma,$$

where

$$\eta_\alpha = \eta_\alpha(u) = \hat{T}^{x_3 - x}(k_\alpha u) - \hat{T}^{x_3 - x}(k_\alpha) u, \quad \alpha = 1, 2.$$

The functionals $\eta_\alpha(u)$ vanish for polynomials of degree zero.

We show that there is a nonuniform mesh $\tilde{\omega}$, such that $\eta_\alpha$ vanishes for first degree polynomials, which must guarantee second order accuracy. To be specific, we consider the functional $\eta_1$. For $u \equiv x_2$ we have

$${\eta_1(x_2)} = \left( \tilde{h}_2 h_2 \right)^{-1} \int_{x_2 - h_2}^{x_2} (\xi - x_2 - h_2)(\xi - x_2) k_1(\xi) d\xi$$

$$+ \left( \tilde{h}_2 h_2^+ \right)^{-1} \int_{x_2}^{x_2 + h_2^+} (x_2 + h_2^+ - \xi)(\xi - x_2) k_1(\xi) d\xi$$

$$= \tilde{h}_2^{-1} \int_0^1 (1 - s) [\left( h_2^+ \right)^2 k_1(x_2 + sh_2^+) - h_2^+ k_1(x_2 - sh_2^+)] ds$$

$$\equiv \tilde{h}_2^{-1} p(h_2, h_2^+).$$

We assume for simplicity that $k_1(x_2)$ is a monotonically increasing function. Then $p(h_2, 0) < 0$ and $p(h_2, h_2) \geq 0$. From this it follows that for a given $h_2$ there is a $\xi = h_2^+ \in (0, h_2)$, which is a solution of the equation $p(h_2, \xi) = 0$, such that $\eta_1 = 0$ for first degree polynomials. Thus, starting with any $h_{21}$, we find successively $h_{22}, \ldots, h_{2N_2}$. For $h_{21} = 1/N_2$ we have $\Sigma^{N_2} h_2 j < 1$; if we take $h_{21} = 1$, then $\Sigma^{N_2} h_2 j = 1$. Consequently,
there is a step $h_{21}$ such that $\Sigma_{1}^{N_{2}}h_{2j} = 1$. In addition, we have

$$1/N_{2} < h_{21} < c_{2}/N_{2}c_{1}, \quad c_{1}/c_{2}N_{2} < h_{2N_{2}} < 1/N_{2}.$$ 

Thus, $\hat{\omega}$ is constructed. The mesh $\hat{\omega}$ for the functional $\eta_{2}$ is constructed analogously.

It is not hard to see that the operators $b_{\alpha}^{-1}A_{\alpha}$, $\alpha = 1, 2$, commute and are symmetric and negative definite in the sense of the inner product with weight $b_{1}b_{2}$. Taking into account property (4) of the operators $\hat{F}^{x_{a}}$, we find that the solution $z$ satisfies (10). Applying the technique developed above, we obtain

**Theorem 2.** Suppose conditions (15) are satisfied for problem (14). Then there is a nonuniform mesh $\hat{\omega}$ for which the difference scheme (16) has second order accuracy; that is, (1) holds with $s = 0$ and $k = 2$.

**Remark 6.** If $q_{a}(x_{a}) \equiv 0$ in (6), then this problem reduces to (14) by means of a simple change of variables.

**Theorem 3.** If conditions (15) are satisfied and if, in addition, $k_{a}(x_{3} - x_{a}) \in W_{o\infty}^{1}(\Omega)$, then (16) has second order accuracy on the uniform mesh $\omega$.

In order to prove this we need to estimate the functionals $\eta_{a}$ on the uniform mesh. To be specific, we consider $\eta_{1}(u)$. We have

$$\eta_{1}(u) = h_{2}^{2}\int_{x_{2} - h_{2}}^{x_{2}}(\xi - x_{2} + h_{2})k_{1}(\xi)\int_{x_{2}}^{\xi}d\xi d\xi$$

$$+ h_{2}^{2}\int_{x_{2} - h_{2}}^{x_{2}}(\xi - x_{2} + h_{2})k_{1}(\xi)\int_{x_{2}}^{\xi}d\xi d\xi$$

$$= h_{2}^{2}\int_{x_{2} - h_{2}}^{x_{2}}(\xi - x_{2} + h_{2})k_{1}(\xi)\int_{x_{2}}^{\xi}[d\xi d\xi$$

$$- \frac{1}{h_{1}h_{2}}\int_{x_{1} - h_{1}}^{x_{1}}\int_{x_{2} - h_{2}}^{x_{2}}\frac{\partial u}{\partial r}(r, p) dr dp] d\xi d\xi$$

$$+ h_{2}^{2}\int_{x_{2} - h_{2}}^{x_{2}}(\xi - x_{2} + h_{2})k_{1}(\xi)\int_{x_{2}}^{\xi}[d\xi d\xi$$

$$- \frac{1}{h_{1}h_{2}}\int_{x_{1} - h_{1}}^{x_{1}}\int_{x_{2} - h_{2}}^{x_{2}}\frac{\partial u}{\partial r}(r, p) dr dp\eta_{1}(x_{2}).$$

Since

$$|\eta_{1}(x_{2})| \leq h_{2} \int_{0}^{1}(1 - s)s \left| k_{2}(x_{2} + sh_{2}) - k_{2}(x_{2} - sh_{2}) \right| ds$$

$$\leq Mh_{2}^{2}\|k_{2}\|_{W_{2}^{2}(\omega)},$$

from the preceding relation we obtain

$$|\eta_{1}(u)| \leq M |h|^{2} (h_{1}h_{2})^{-1/2} \|u\|_{W_{2}^{2}(\omega)},$$

where $e = (x_{1} - h_{1}, x_{1} + h_{1}) \times (x_{2} - h_{2}, x_{2} + h_{2})$. An analogous estimate also holds for $\eta_{2}(u)$. Substituting these estimates in (10), we obtain the required result.
From what has been said above, we conclude that for "bad" properties of the coefficients $k_a$, the scheme (16) has second order accuracy on a special nonuniform mesh. If the coefficients are "smoother", then second order accuracy is obtained also on a uniform mesh.

§6. A fourth order equation

Compatible estimates of type (1) can also be obtained for some problems for a fourth order equation. We consider the second boundary value problem for the fourth order equation with variable coefficients

$$\sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i^2} \left( a_{ij}(x) \frac{\partial^2 u}{\partial x_i^2} \right) = f(x), \quad x \in \Omega,$$

$$u(x) = 0, \quad \frac{\partial^2 u(x)}{\partial n^2} = 0, \quad x \in \Gamma,$$

where $n$ is the outward normal to the boundary $\Gamma$. We cover the plane $(x_1, x_2)$ with the mesh $E_h = \{x = (x_1, x_2): x_a = i_a h_a, i_a = 0, \pm 1, \ldots, h_a = l_a/N_a, \alpha = 1, 2\}$ and introduce the notation $\omega = \Omega \cap E_h, \bar{\omega} = \bar{\Omega} \cap E_h, \gamma = \bar{\omega} \setminus \omega$, and $\gamma_{\pm a} = \{x \in \gamma: \cos(x_a, n) = \pm 1\}, \alpha = 1, 2$. We approximate (17) by the difference scheme

$$\sum_{i,j=1}^{2} \left( T_{3-i}^2(a_{ij}) y_{x_i x_j} \right) \xi_i \xi_j = T_2^2 T_2^2 f, \quad x \in \omega,$$

$$y(x) = 0, \quad x \in \gamma, \quad y_{\gamma a} x = 0, \quad x \in \gamma_{\pm a}, \alpha = 1, 2.$$

If $f(x) \in W_2^{-l}(\Omega)$ and $\Sigma_{i,j=1}^{2} a_{ij} \xi_i \xi_j \geq c_0 (\xi_1^2 + \xi_2^2)$, where $c_0 > 0$ and $a_{ij}(x) \in W_2^l(\Omega) \cap W_2^{l+1}(\Omega), l = 0, 1$, then for (18) we obtain (1) with $s = 2$ and $k = 3 + l$.

Analogous results are also obtained for a quasilinear fourth order equation.

§7. Parabolic equations

We apply the method developed above to the construction and study of the convergence of a difference scheme for parabolic equations. We consider the problem

$$\frac{\partial u}{\partial t} = Lu + f(x, t), \quad (x, t) \in Q_T = \Omega \times [0, T],$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad u(x, t) = 0, \quad x \in \Gamma, 0 < t < T,$$

where the operator $L$ is defined by (6). For this problem we construct the purely implicit difference scheme

$$\begin{align*}
\begin{array}{ll}
b_1 b_2 y_i = \Lambda y + \varphi, & \quad x \in \omega, t \in \omega_r, \\
y(x, 0) = T^x T^{x^t} u_0(x), & \quad x \in \omega, \\
y(x, t) = 0, & \quad x \in \gamma, t \in \omega_r, \\
y = y^j = y(x, t_j), & \quad y_i = (y^j - y^{j-1})/\tau, & \quad j = 1, 2, \ldots, J,
\end{array}
\end{align*}$$

\[\text{where } \omega_r = \{t_j = j \tau, j = 1, \ldots, J, \tau = T/J \} \text{ and the remaining notation is the same as in } \S3. \text{ In what follows we study the convergence of (20) for generalized solutions in } W_2^{l+1}(Q_T).

We introduce the Steklov averaging operator $T_0$ with respect to one variable

$$T_0 u(x, t) = \frac{1}{\tau} \int_{t-\tau}^{t} u(x, \xi) \, d\xi, \quad t \in \omega_r, \quad T_0 u(x, t) = u_0(x), \quad t = 0,$$

and set $\bar{u}(x, t) = T_0 \bar{u}(x, t)$, where the average $\bar{u}(x, t)$ is that defined in $\S3$.

The following proposition holds.
THEOREM 4. Suppose the solution $u(x, t)$ of (19) belongs to $W^{1,1}(Q_T)$. Then the solution of (20) converges in the norm of $L_2(\omega \times \omega)$ to the average $\overline{u}$ of the given solution $u$, and
\[
\| y - \overline{u} \|_{L_2(\omega \times \omega)} \leq M \left( \tau \frac{\partial u}{\partial t} \right)_{L_2(Q_T)} + | h | \cdot \| u \|_{W_2^{1,0}(Q_T)}
\]
where the constant $M$ is independent of $\tau$ and $h$.

PROOF. For the function $z = y - \overline{u}$ we obtain the problem
\[
b_1 b_2 z_t - \Delta z + \Psi, \quad x \in \omega, \quad z(x, t) = 0, \quad x \in \gamma, \quad t \in \omega_t, \quad z(x, 0) = T^{\gamma} T^{\omega} u_0(x) - b_1 b_2 \overline{u}(x, t) \big|_{t=0}, \quad x \in \omega.
\]
(22)
The error $\Psi$ of the approximation can be represented in the form
\[
\Psi = \eta_0 + \Lambda_1 \eta_1 + \Lambda_2 \eta_2,
\]
where
\[
\eta_0(x, t) = T^{\gamma} T^{\omega} u(x, t) - b_1 b_2 \overline{u}(x, t), \quad x \in \omega, t \in \omega_t,
\]
\[
\eta_1(x, t) = b_2 u - T_0 T^{\omega} (u), \quad \eta_2(x, t) = b_1 u - T_0 T^{\omega} (u).
\]
The solution of (22) satisfies the a priori estimate
\[
\| z \|_{L_2(\omega \times \omega)} \leq M \sum_{\alpha=0}^{2} \| \eta_\alpha \|_{L_2(\omega \times \omega)}.
\]
This inequality is proved in two stages. First, we use the method of isolating stationary inhomogeneities (see [2]) to derive an a priori estimate connected with the error of the elliptic part of the operator. Then we estimate the solution of the difference scheme with right-hand side $T_1 \eta_1$ (see [8]).

To obtain (21) it suffices to estimate the $L_2$-norm of $\eta_\alpha$, $\alpha = 1, 2$. The expressions $\eta_\alpha$ are linear bounded functionals in $W^{1,1}(Q_T)$ which vanish for a constant. Using the technique developed in [8], from (23) we obtain the desired estimate (21).

REMARK 7. An analogous result can also be proved for a scheme with weights. In particular, for $\tau \leq 2/\| \Lambda \|$ we obtain an estimate of the convergence of an explicit scheme.

REMARK 8. The convergence estimate is proved for arbitrary $\tau$ and $| h |$. Hence, passing to the limit in (21) as $\tau \to 0$, we obtain an estimate for the convergence rate of the method of lines for problem (19) of order $O(| h |)$ for solutions in $W^{1,1}(Q_T)$.

REMARK 9. The results of §5 carry over to parabolic problems of the form (19).

REMARK 10. The estimates obtained above also hold in the case when the right-hand side $f$ depends on $u$ and satisfies a Lipschitz condition with respect to this variable.

REMARK 11. All our results also remain valid in the three-dimensional case.

REMARK 12. We can obtain an estimate of the convergence of the difference scheme (20) under weaker assumptions on the smoothness of the solution $u(x, t)$ with respect to $t$. Following [15], we denote by
\[
\tau_1(v, \delta)_{L_2} = \left( \int_0^T \sup_{0 \leq \eta \leq \delta} \| v(x, t) - v(x, t - \eta) \|^2_{L_2(\Omega)} dt \right)^{1/2}
\]
the integral $\tau_1$-modulus of continuity, with respect to $t$, of the function $v(x, t)$. In this case it turns out that if $u \in W^{1,0}_2(Q_T)$, then
\[
\| y - \overline{u} \|_{L_2(\omega \times \omega)} \leq M \left[ \tau_1(u, \tau)_{L_2} + | h | \cdot \| u \|_{W_2^{1,0}(Q_T)} \right].
\]
If $\frac{\partial u}{\partial t} \in L_2(Q_T)$, this yields (21). But here we obtain convergence under weaker assumptions on the smoothness of the desired solution (in particular, when $u(x, t)$ has bounded variation in $t$; see [15]).

BIBLIOGRAPHY


Translated by C. CONSTANDA

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*Editor’s note. The original combines the title of b) with the journal information of a.*