**Math151 Lecture Notes**
**6.3 - The Definite Integral**

In section 6.2 we found that the area under $f(x) = x^2 + 1$ on $[0, 2]$ was 14/3 square units. We found this by calculating

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2}{n}\left(\left(\frac{2i}{n}\right)^2 + 1\right)\right] = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 + 1\right]$$

We can now use $\int$, call an *integral sign*, to rewrite the above limit. This symbol was founded by Leibniz and was appropriately chosen because the elongated $S$ represents a limit of sums.

**Theorem**
If $f$ is integrable on $[a, b]$, then

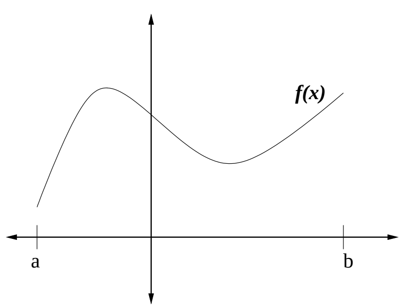
$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

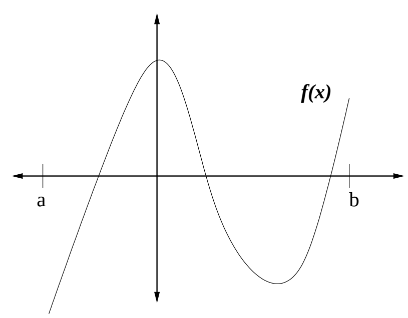
Example 1: Rewrite $\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[ \left( \frac{2i}{n} \right)^2 + 1 \right]$ as an integral on [0,2].

## Geometric Intrepretation of an Integral

$$\int_a^b f(x)dx = A_1 - A_2$$

where $A_1$ is the area of the region above the $x$-axis and below the graph of $f(x)$ and $A_2$ is the area of the region below the $x$-axis and above the graph of $f(x)$



Example 2: Express each limit as a definite integral.

(a) $\lim_{||P||\to 0} \sum_{i=1}^{n} \sqrt{x_i}\, \Delta x_i$ on [1,4]

(b) $\lim_{||P||\to 0} \sum_{i=1}^{n} \cos(x_i)\Delta x_i$ on $[0, \pi]$

(c) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2}$

(d) $\lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^{n} \left[\left(\frac{\pi i}{n}\right)^3 + \left(\frac{\pi i}{n} \sin\left(\frac{\pi i}{n}\right)\right)\right]$

Example 3: Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1-x^2}dx$ (b) $\int_0^3 (x-1)dx$

**Midpoint Rule**

$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \Delta x[f(\overline{x}_1) + ... + f(\overline{x}_n)]$ where $\Delta x = \dfrac{b-a}{n}$ and $\overline{x}_i = \dfrac{1}{2}(x_{i-1} + x_i) =$ midpoint of $[x_{i-1}, x_i]$

Example 4: Use midpoint to approximate $\int_0^2 x^2 + 1dx$ with $n = 5$

**• Definition**

If $a > b$, then $\int_a^b f(x)dx = -\int_b^a f(x)dx$

If $a = b$, then $\int_a^a f(x)dx = 0$

**• Properties of the Integral**

Suppose that all of the following integrals exist.

If $c$ is any constant, then

1. $\int_a^b c dx = c(b-a)$
2. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x) + \int_a^b g(x)$
3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
4. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x) - \int_a^b g(x)$
5. $\int_a^b f(x)dx = \int_a^d f(x)\ dx + \int_d^b f(x)dx$

**• The following properties are only true if** $a \leq b$

6. If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)dx \geq 0$
7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$
9. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

Example 5: Use Property 8 to estimate the value of $\int_1^4 \sqrt{x}dx$

Example 6: Show that $\int_1^4 \sqrt{1+x^2}dx \geq 7.5$