Uniformly Bounded Spaces

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1 Introduction

The first step in the geometric and asymptotic group theory is transforming a finitely generated group into *a Cayley graph* or into *a word metric space*. The Cayley graph of a group G with a fixed generating set $S = S^{-1}$ is the graph with a set of vertices G and a set of edges $\{(g, h) : g^{-1}h \in S\}$. The word metric space is the space with a set of points G and with distance between two points g, h equal to the length of the shortest presentation of the element $g^{-1}h$ as a product of elements from S (see [G]).

These two ways of geometrization are essentially equivalent. But one of the problems is dependence of the constructed geometrical objects on the choice of the generating set S. When we change the generating set S, then the word metric space changes to a metric space bi-Lipschitz equivalent to the original.

Hence, the bi-Lipschitz equivalence and a more general notion of quasi-isometry become the main equivalence relations of metric spaces in the asymptotic (large scale) geometry. Quasi-isometric spaces from the large scale point of view are identical. Thus, the notion of metric space, as a tool of the group theory, bears some superfluous structure.

In the paper we introduce a notion of a uniformly bounded space. We hope that this notion is an appropriate structure resembling the large scale geometry of groups and metric spaces. And in some sense, this notion is a "pure" structure. The Cayley u.b. space of a group is defined without using any generating sets and depends on the group only. Thus, we propose a method of geometrization of the infinitely generated groups. In the case of finitely generated groups the Cayley graphs are bi-Lipschitz equivalent if and only if the corresponding Cayley u.b. spaces are equal.

On the other hand, the proposed structure is sufficiently rich. As an argument, we show that the theory of non-amenable metric spaces, groups (not only finitely generated) and pseudogroups (see [CGH, HS]) can be developed for the u.b. spaces.

The structure of our paper is the following.

In the section "General Definitions" we define pseudoequivalences on sets and formulate some technical definitions and propositions. A u.b. space is a set with a given pseudoequivalence on it. Pseudoequivalence is a kind of a fuzzy equivalence (a natural prototype of pseudoequivalence is the relation "a relative" between people), and formalizes, in a special way, the notion "to be close". Comparing with the classical formalizations (topological space, uniform topological space [Eng, B]), our notion is adopted for investigation of the large scale properties of metric spaces.

In the section "Some Classical Examples" we construct standard u.b. spaces associated to graphs, metric spaces and groups. We associate to any discrete group its *Cayley space*. It is a direct generalization of the word metric spaces and pretends to be their quintessence.

The section "Morphisms of Uniformly Bounded Spaces" defines analogues of Lipschitz on a large scale maps, bi-Lipschitz equivalences and quasi-isometries. The section "Nets" defines the notion of a cobounded set and shows their relation to the large scale morphisms.

In the section "Group Actions on Uniformly Bounded Spaces" we investigate representations of a group by large scale isomorphisms of a u.b. space. Action of a group on a u.b. space defines some pseudoequivalence on the group. If the action is *quasi-transitive* (cobounded) and discrete, then the induced u.b. space is l.s. isomorphic to the underlying space and coincides with the Cayley space of the group. This generalizes the classical facts about proper cocompact actions of a group on a manifold (see [G]).

Another interesting topic is an action by shifts. *Shift* is a (partial) map on a finite distance from the identity map. Spaces from an important class of *regular (uniformly discrete)* u.b. spaces can be completely described by their pseudogroups of shifts. An exposition of the pseudogroup theory, closely related to the topics of our paper, you can find in [CGH].

The last section "Measures of Uniformly Bounded Spaces" develops a theory of uniformly bounded measures. Uniformly bounded measures in the discrete (regular) case are uniquely defined by a positive bounded function — the *distribution* of the measure. Thus, in this case the theory of u.b. measures is just a theory of the bounded positive functions on the space. Since we are interested in large scale properties of the u.b. spaces, we have to identify shift equivalent sets (sets A and B are shift equivalent if there exists a bijective shift carrying A to B) and shift equivalent measures. This identification leads us to a semigroup $\mathcal{L}^+_\infty(\mathfrak{X})$ of the equivalence classes of the u.b. measures. The semigroup $\mathcal{L}^+_\infty(\mathfrak{X})$ is invariant under the large scale isomorphisms.

We investigate the semigroup properties of $\mathcal{L}^+_\infty(\mathfrak{X})$. One of the interesting questions is existence of a nontrivial homomorphism $m : \mathcal{L}^+_\infty(\mathfrak{X}) \to \mathbb{R}^+$ of $\mathcal{L}^+_\infty(\mathfrak{X})$ onto the semigroup $\mathbb{R}^+$ of positive reals. The u.b. space for which such a homomorphism exists are called *amenable*.

The theory of amenable regular u.b. spaces unites the classical theories of amenable groups, pseudogroups and metric spaces into a natural fine framework.

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2 General Definitions

2.1. Recall that *relation* on a set $\mathfrak{X}$ is a subset of the Cartesian product $\mathfrak{X} \times \mathfrak{X}$. Product of relations $\mathcal{R}_1$ and $\mathcal{R}_2$ is the relation $\mathcal{R}_1 \cdot \mathcal{R}_2$ defined by the rule:

$$(x, y) \in \mathcal{R}_1 \cdot \mathcal{R}_2 \iff \exists z \in \mathfrak{X} : (x, z) \in \mathcal{R}_1 \text{ and } (z, y) \in \mathcal{R}_2.$$

A relation $\mathcal{R}$ on a set $\mathfrak{X}$ is *reflexive* if for any $x \in \mathfrak{X}$ we have $(x, x) \in \mathcal{R}$.

If $\mathcal{R}$ is a relation on a set $\mathfrak{X}$, then *the inverse relation* is the relation $\mathcal{R}^-$ defined by the rule:

$$(x, y) \in \mathcal{R}^- \iff (y, x) \in \mathcal{R}.$$

Definition. *Pseudoequivalence on a set* $\mathfrak{X}$ is a nonempty set Γ of reflexive relations on the set $\mathfrak{X}$, satisfying the following conditions:

1. If $\mathcal{R} \in \Gamma$, then $\mathcal{R}^- \in \Gamma$.
2. If $\mathcal{R}_1, \mathcal{R}_2 \in \Gamma$, then $\mathcal{R}_1 \cdot \mathcal{R}_2 \in \Gamma$.
3. If $\mathcal{R} \in \Gamma$ and $\mathcal{R}' \subseteq \mathcal{R}$, then $\mathcal{R}' \in \Gamma$.

Uniformly bounded space is the set $\mathfrak{X}$ with a given pseudoequivalence $\Gamma(\mathfrak{X})$ on it. The elements of pseudoequivalence are called *uniformly bounded relations* on the u.b. space $\mathfrak{X}$.

2.2. From the definition follows that *the zero* relation (*the diagonal relation*)

$$\mathcal{O} \stackrel{\text{def}}{=} \{(x, x) : x \in \mathfrak{X}\}$$

belongs to every pseudoequivalence on the set $\mathfrak{X}$, since it is a subrelation of every reflexive relation on $\mathfrak{X}$. Since for reflexive relations $\mathcal{R}_1, \mathcal{R}_2$ we have $\mathcal{R}_1 \cup \mathcal{R}_2 \subseteq \mathcal{R}_1 \cdot \mathcal{R}_2$, the union of any two elements of a pseudoequivalence belongs to the pseudoequivalence.

By analogy with the notations of the classical theory of uniform topological spaces [Eng], instead of $(x, y) \in \mathcal{R}$ we write $d_{\mathfrak{X}}(x, y) \le \mathcal{R}$ or $d(x, y) \le \mathcal{R}$. And if $\mathcal{R}_1 \subseteq \mathcal{R}_2$, then we write $\mathcal{R}_1 \le \mathcal{R}_2$.

2.3. *A ball of a radius* $\mathcal{R}$ with a center in the point $x \in \mathfrak{X}$, where $\mathcal{R} \in \Gamma(\mathfrak{X})$, is the set

$$B_x(\mathcal{R}) \stackrel{\text{def}}{=} \{y : d_{\mathfrak{X}}(x, y) \le \mathcal{R}\}$$

If $F \subseteq \mathfrak{X}$, then *the* $\mathcal{R}$*-neighborhood* of the set A is the set

$$\operatorname{Neib}(A, \mathcal{R}) \stackrel{\text{def}}{=} \bigcup_{x \in A} B_x(\mathcal{R}).$$

2.4. A set $A \subseteq \mathfrak{X}$ is *bounded* if there exists $\mathcal{R} \in \Gamma(\mathfrak{X})$ such that for all $x, y \in A$ we have $(x, y) \in \mathcal{R}$. In this case we write $\mathrm{diam}(A) \leq \mathcal{R}$.

A set $\mathcal{A}$ of subsets of the space $\mathfrak{X}$ is said to be *uniformly bounded* if there exists $\mathcal{R} \in \Gamma(\mathfrak{X})$ such, that for any $A \in \mathcal{A}$ we have $\mathrm{diam}(A) \leq \mathcal{R}$.

2.5. Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be a subset of the u.b. space $\mathfrak{X}$. If $\mathcal{R}$ is a relation on $\mathfrak{X}$, then *the induced relation* $\mathcal{R}|_{\mathfrak{Y}}$ is the relation $\mathcal{R} \cap \mathfrak{Y} \times \mathfrak{Y}$. Obviously, the set of the relations on $\mathfrak{Y}$, induced by the uniformly bounded relations on the u.b. space $\mathfrak{X}$, is a pseudoequivalence on $\mathfrak{Y}$. The set $\mathfrak{Y}$ together with this pseudoequivalence is called *a subspace* of the u.b. space $\mathfrak{X}$.

2.6. A u.b. space is *connected* if any its finite subset is bounded. Obviously, a space is connected iff the set $\{x_1, x_2\}$ is bounded for any $x_1, x_2 \in \mathfrak{X}$.

We can introduce a notion of connectivity components for a u.b. spaces by the following construction.

Two points $x, y \in \mathfrak{X}$ are said to be at a finite distance (we use the notation: $d(x, y) < \infty$), if for some $\mathcal{R} \in \Gamma(\mathfrak{X})$ we have $d(x, y) \leq \mathcal{R}$.

One can see that the relation "to be on a finite distance" is an equivalence relation on the space $\mathfrak{X}$. The equivalence classes of this relation are the connectivity components of the space $\mathfrak{X}$. The components of a u.b. space are its maximal connected subspaces.

2.7. Another important notions are the notions of a base and a prebase of a u.b. space.

Definition. A subset Γ' of the pseudoequivalence on a u.b. space $\mathfrak{X}$ is *a base of the u.b. space (a base of the pseudoequivalence)* if any uniformly bounded relation on the space is a subrelation of some element of Γ'.

The intersection of any set of pseudoequivalences is a pseudoequivalence. The intersection of all the pseudoequivalences, containing a given set Γ_0 of reflexive relations, is *the pseudoequivalence generated by the set* Γ_0. In this case, the set Γ_0 is called *a prebase* of the generated u.b. space.

Proposition 2.1. *Let Γ_0 be a set of symmetrical reflexive relations on the set $\mathfrak{X}$. A reflexive relation $\mathcal{R}$ belongs to the pseudoequivalence Γ, generated by the set Γ_0 iff the relation $\mathcal{R}$ is a subrelation of a relation $\mathcal{R}_1 \cdot \mathcal{R}_2 \cdots \mathcal{R}_n$, where $\mathcal{R}_i \in \Gamma_0$.*

Uniformly bounded space is *monogenic*, if it has a prebase consisting of a single relation.

3 Some Classical Examples

3.1 Graphs

The *standard pseudoequivalence* on the set of vertices of a graph is the pseudoequivalence, generated by the relation "to be connected by an edge".

Thus, according to the definition, any graph as a standard u.b. space is monogenic. It is easy to see, that the converse statement is also true.

Proposition 3.1. *Let $\mathfrak{X}$ be a monogenic u.b. space. Then there exists a graph with the set of vertices $\mathfrak{X}$ for which the standard pseudoequivalence coincides with the pseudoequivalence of the space $\mathfrak{X}$.*

Obviously, a graph is connected if and only if it is connected as the standard u.b. space.

3.2 Metric Spaces

Let $\mathfrak{X}$ be a metric space. Distance between the points $x, y \in \mathfrak{X}$ will be denoted $|x-y|_{\mathfrak{X}}$ or just $|x-y|$.

Let $\mathcal{R}$ be a relation on $\mathfrak{X}$. Let

$$|\mathcal{R}| \stackrel{\text{def}}{=} \sup_{d(x,y)\leq\mathcal{R}} |x-y|$$

be *the norm* of the relation.

Let $\Gamma(\mathfrak{X})$ be the set of all reflexive relations with finite norm on the metric space $\mathfrak{X}$. It is a simple exercise to check that $\Gamma(\mathfrak{X})$ is a pseudoequivalence on $\mathfrak{X}$.

This pseudoequivalence is called *the standard pseudoequivalence of the metric space* $\mathfrak{X}$, and the u.b. space defined by this pseudoequivalence is called *the (standard) u.b. space of the metric space* $\mathfrak{X}$.

The standard pseudoequivalence of the metric space $\mathfrak{X}$ has a base consisting of the relations $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$, where $d(x,y) \leq \mathcal{R}_n$ iff $|x-y| \leq n$.

Note, that the standard u.b. space of a metric space is always connected.

In fact, the notion of a u.b. space was specially developed to formalize the large scale properties of metric spaces. The original notions for metric spaces coincide with the notions for the standard u.b. space; i.e. a set (a set of sets) is bounded (is uniformly bounded) in a metric space if and only if it is bounded (uniformly bounded) in the standard u.b. space of the metric space.

3.3 Quasi-geodesic Metric Spaces

Any connected graph is, in a natural way, a metric space with distance between two vertices equal to the shortest length of a chain connecting these vertices.

Proposition 3.2. *The standard pseudoequivalence of a connected graph coincides with the standard pseudoequivalence of this graph as a metric space.*

Definition. A space $\mathfrak{X}$ is quasi-geodesic (compare with [G, N1]) if there exist positive constants C, L, δ such that for any two points $x_1, x_2 \in \mathfrak{X}$ there exists a sequence of points $x_1 = p_0, p_1, \ldots, p_n = x_2 \in \mathfrak{X}$, where $n < L|x_1 - x_2| + \delta$ and $|p_i - p_{i+1}| < C$. The number C is called *the quasi-geodesity radius*, and the sequence is called *the quasi-geodesic*.

The following simple theorem characterizes the standard pseudoequivalences of quasi-geodesic spaces.

Theorem 3.3. *Any graph is a quasi-geodesic metric space.*

The standard pseudoequivalence of a quasi-geodesic metric space $\mathfrak{X}$ coincides with the standard pseudoequivalence of some graph with the set of vertices $\mathfrak{X}$.

Proof. Any graph is quasi-geodesic with the corresponding constants $C = 1, L = 1, \delta = 0$.

If $\mathfrak{X}$ is quasi-geodesic, then construct a graph with the set of vertices $\mathfrak{X}$ and connect two points $x, y \in \mathfrak{X}$ with an edge if and only if $|x-y| < C$, where C is the quasi-geodesity radius. □

The following theorem is a corollary of 3.1 and 3.3.

Theorem 3.4. *A pseudoequivalence is the standard pseudoequivalence of a quasi-geodesic metric space iff it is connected and monogenic.*

Open Questions. **1.** When is a u.b. space $\mathfrak{X}$ the standard u.b. space of a metric space with the set of points $\mathfrak{X}$?

2. We can generalize the notion of a metric space, defining metric as a map $|\cdot,\cdot| : \mathfrak{X} \times \mathfrak{X} \to \mathcal{H}$, where $\mathcal{H}$ is an ordered semigroup (*the semigroup of distances*), with the usual axioms. We define the standard u.b. spaces of such metric spaces in the similar way as for the usual metric spaces.

It seems, that any u.b. space is a standard u.b. space of a metric space in this general meaning. What ordered semigroups appear as a semigroup of distances?

We can define a more exotic kind of pseudoequivalences on metric spaces. Let x_0 be a fixed point of $\mathfrak{X}$. For a given relation $\mathcal{R}$ on a metric space $\mathfrak{X}$ we define a function

$$D_{\mathcal{R}}(R) = \sup\{|x - y| : (x, y) \in \mathcal{R} \cup \mathcal{R}^-, |x - x_0| < R\}.$$

An $O(R)$-pseudoequivalence is the set of all reflexive relations $\mathfrak{X}$ for which

$$\limsup_{R\to\infty} \frac{D_{\mathcal{R}}(R)}{R} < \infty,$$

and *an $o(R)$-pseudoequivalence* is the set of all reflexive relations on $\mathfrak{X}$ for which

$$\lim_{R\to\infty} \frac{D_{\mathcal{R}}(R)}{R} = 0.$$

One can see that the defined pseudoequivalences do not depend on the choice of the point x_0, and that they are really pseudoequivalences on $\mathfrak{X}$.

Similarly, one can define the pseudoequivalences $O(F(R))$ and $o(F(R))$ for some other functions $F : \mathbb{R}^+ \to \mathbb{R}^+$.

Open Question. It is an interesting problem to investigate the $O(R)$-pseudoequivalences. One of the questions is to find a criterion for a $O(R)$-u.b. space to be virtually regular (see a definition below.) Metric spaces with virtually regular $O(R)$-u.b. space are closely related to the spaces of polynomial growth. At least, the groups with virtually regular $O(R)$-u.b. space of the word metric space are exactly the groups of polynomial growth.

3.4 Discrete Groups

Let G be a group. If $\mathcal{R}$ is a reflexive relation on G, then we define

$$D(\mathcal{R}) \stackrel{\text{def}}{=} \{g^{-1}h : (g, h) \in \mathcal{R}\}.$$

The set of all reflexive relations $\mathcal{R}$ on G with finite $D(\mathcal{R})$ is a pseudoequivalence. This pseudoequivalence we call *the Cayley pseudoequivalence* on the group G. The constructed u.b. space is called *the (left-invariant) Cayley space of the group G*.

It is easy to see that the Cayley pseudoequivalence of the group has a base

$$\{\mathcal{R}_B : d(g, h) \le \mathcal{R}_B \iff g^{-1}h \in B, |B| < \infty\}.$$

The Cayley pseudoequivalence is always connected.

Theorem 3.5. *The Cayley pseudoequivalence of a group G is monogenic iff the group is finitely generated. If the group is finitely generated, then the Cayley pseudoequivalence coincides with the standard pseudoequivalence of the word metric space of the group and with the standard pseudoequivalence of the Cayley graph.*

We can also define a right-invariant Cayley pseudoequivalence of the group putting $D(\mathcal{R}) \stackrel{\text{def}}{=} \{gh^{-1} : d(g, h) \le \mathcal{R}\}$. In general these pseudoequivalences do not coincide. More exactly, the following theorem holds.

Proposition 3.6. *The left-invariant Cayley pseudoequivalence of a group coincides with the right-invariant pseudoequivalence iff the group is an FC-group (i.e. if all its conjugacy classes are finite).*

Open Question. It is interesting to investigate "the distortion" of the left-invariant Cayley pseudoequivalence in the pseudoequivalence generated by the union of the left-invariant and the right-invariant pseudoequivalences (*the bi-invariant pseudoequivalence*), i.e. how large is the second pseudoequivalence in comparison with the first.

Another open question is to describe the groups with virtually regular bi-invariant pseudoequivalence.

4 Morphisms of Uniformly Bounded Spaces

4.1 Maps

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be uniformly bounded spaces.

A map $f : \mathfrak{X} \to \mathfrak{Y}$ is *a morphism of the uniformly bounded spaces* if for any $\mathcal{R} \in \Gamma(\mathfrak{X})$ the relation $f(\mathcal{R}) \cup \mathcal{O}$ is uniformly bounded. (The relation $f(\mathcal{R})$ is the image of the subset $\mathcal{R} \subseteq \mathfrak{X} \times \mathfrak{X}$ under the map $f \times f$.)

A bijection $f : \mathfrak{X} \to \mathfrak{Y}$ is *an isomorphism of the u.b. spaces* if f and f^{-1} are morphisms.

In the case of the standard pseudoequivalences of quasi-geodesic metric spaces, a map is a morphism of the u.b. spaces if and only if it is *Lipschitz on a large scale* (see the definition below, and [G]).

Definition. A map $f : \mathfrak{X} \to \mathfrak{Y}$ is said to be L-Lipschitz on the δ -scale if

$$|f(x_1) - f(x_2)|_{\mathfrak{Y}} < L|x_1 - x_2|_{\mathfrak{X}} + \delta$$

for all $x_1, x_2 \in \mathfrak{X}$.

In the general situation, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of the standard u.b. spaces of the metric spaces $\mathfrak{X}$ and $\mathfrak{Y}$ iff there exists a function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any two points $x_1, x_2 \in \mathfrak{X}$ we have:

$$|f(x_1) - f(x_2)|_{\mathfrak{Y}} \leq L(|x_1 - x_2|_{\mathfrak{X}}).$$

4.2 Large Scale Morphisms

For a good asymptotic theory of u.b. spaces, we have to identify *shift equivalent* morphisms, as it is done in the classical situation for metric spaces (see [G]).

Two maps $f_1, f_2 : \mathfrak{X} \to \mathfrak{Y}$ are *shift equivalent* ($f_1 \approx f_2$) if there exists $\mathcal{R} \in \Gamma(\mathfrak{Y})$ such that for any $x \in \mathfrak{X}$ we have $d(f_1(x), f_2(x)) \leq \mathcal{R}$.

The maps which are shift equivalent to the identity are called *shifts*.

The uniformly bounded spaces together with the classes of shift equivalent morphisms form *a large scale category* LS of uniformly bounded spaces (compare with [G, N1]).

The isomorphisms of this category are called *the large scale isomorphisms (l.s. isomorphism)*.

In the standard pseudoequivalence of quasi-geodesic metric spaces the notion "l.s. isomorphism" coincides with the notion "quasi-isometry" (see [G]).

Open Question. What group properties are invariant under (l.s.) isomorphisms of the Cayley spaces? Are the properties of a group to be an FC-group, FC-hypercentral or nilpotent, l.s. invariant?

5 Nets

5.1. A subset $N \subseteq \mathfrak{X}$ of a uniformly bounded space is *a net (is cobounded)* if there exists an $\mathcal{R} \in \Gamma(\mathfrak{X})$, such that $\mathrm{Neib}(N, \mathcal{R}) = \mathfrak{X}$. In this case we say that N is an $\mathcal{R}$-net.

An Equivalent Definition. A subset $N \subseteq \mathfrak{X}$ is a net if the embedding $\mathrm{id} : N \hookrightarrow \mathfrak{X}$ is an l.s. isomorphism.

5.2. A set $A \subseteq \mathfrak{X}$ is said to be $\mathcal{R}$-separated if for any two points $x, y \in A$ we have $d(x, y) \nleq \mathcal{R}$.

Any maximal $\mathcal{R}$-separated subset of $\mathfrak{X}$ is an $\mathcal{R}$-net. Such a maximal set always exists, due Zohrn lemma.

Proposition 5.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an l.s. isomorphism of the u.b. spaces. There exist nets $N_{\mathfrak{X}} \subseteq \mathfrak{X}, N_{\mathfrak{Y}} \subseteq \mathfrak{Y}$ such that the map $f : N_{\mathfrak{X}} \to N_{\mathfrak{Y}}$ is an isomorphism of the subspaces.*

Thus, if any net of $\mathfrak{X}$ is isomorphic to $\mathfrak{X}$ and any net of $\mathfrak{Y}$ is isomorphic to $\mathfrak{Y}$, then from a l.s. isomorphism of the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ follows there isomorphism. Bi-Lipschitz equivalence of separated nets in metric spaces was investigated in [P, Bog, N2, N3, W, M, BK].

5.3. A u.b. space is *regular (uniformly discrete)* if for any uniformly bounded relation $\mathcal{R}$ we have:

$$\max_{x \in \mathfrak{X}} |B_x(\mathcal{R})| < \infty.$$

(M.Gromov in [G] uses a term "uniformly quasi-locally bounded space".)

The main example of a uniformly discrete space is the Cayley space of a group.

If the space $\mathfrak{X}$ has a regular $\mathcal{R}$-net, then any its $\mathcal{R}^2$-separated net is regular, and thus the space has many regular nets.

A space is called *virtually regular* if it has a regular net.

6 Group Actions on Uniformly Bounded Spaces

6.1 Basic Definitions

Definition. Let G be a group. A map $\phi : G \to \mathfrak{X}^{\mathfrak{X}}$ is called an action of the group by (uniform) l.s. isomorphisms on a u.b. space $\mathfrak{X}$ if the following conditions hold:

1. For any $\mathcal{R} \in \Gamma(\mathfrak{X}), g \in G$ there exists $\mathcal{R}_g \in \Gamma(\mathfrak{X})$ such that for any $x_1, x_2 \in \mathfrak{X}$ the inequality $d(x_1, x_2) \le \mathcal{R}$ implies

$$d\left(\phi\left(g\right)\left(x_1\right), \phi\left(g\right)\left(x_2\right)\right) \le \mathcal{R}_g.$$

2. There exists $\mathcal{R}_1 \in \Gamma(\mathfrak{X})$ such that

$$d\left(\phi\left(g_1\right)\left(\phi\left(g_2\right)\left(x\right)\right), \phi\left(g_1 g_2\right)\left(x\right)\right) \le \mathcal{R}_1$$

for all $g_1, g_2 \in G, x \in \mathfrak{X}$.

3. There exists $\mathcal{R}_2 \in \Gamma(\mathfrak{X})$ such that

$$\sup_{x \in \mathfrak{X}} d(\phi(1)(x), x) \le \mathcal{R}_2$$

for any $x \in \mathfrak{X}$.

Two actions ϕ_1, ϕ_2 are *shift equivalent* if there exists $\mathcal{R} \in \Gamma(\mathfrak{X})$ such that

$$d(\phi_1(g)(x), \phi_2(g)(x)) \leq \mathcal{R}$$

for all $g \in G, x \in \mathfrak{X}$.

Further, instead of writing $\phi(g)(x)$ we write $g(x)$. We distinguish $\phi(g_1g_2)(x)$ and $\phi(g_1)(\phi(g_2)(x))$ writing $g_1g_2(x)$ in the first case, and $g_1 \circ g_2(x)$ in the second.

If $\mathfrak{X}$ is a quasi-geodesic metric space with the standard pseudoequivalence, then the group G acting on the uniformly bounded space, acts on the metric space by quasi-isometries. A definition of an action by quasi-isometries on metric spaces you can find in [N1]. Most of the definitions and propositions of the [N1] can be generalized for the actions by l.s. isomorphisms on u.b. spaces.

For example:

Definitions. Let a group G be acting on a u.b. space $\mathfrak{X}$ by l.s. isomorphisms.

- The group is *quasi-transitive*, if for some (and thus for all) $x \in \mathfrak{X}$ the orbit Gx of the point x is an $\mathcal{R}$-net for some $\mathcal{R} \in \Gamma(\mathfrak{X})$. One can prove that $\mathcal{R}$ can be chosen independently on x.
- The set $\mathrm{S}(G) \stackrel{\text{def}}{=} \{g \in G : g \approx 1\}$ is a normal subgroup of the group G, and we call it *the shift subgroup*.
- The group G is *discrete*, if for any bounded $B \subseteq \mathfrak{X}$ and for any $x \in \mathfrak{X}$ the set $\{g \in G : g(x) \in B\}$ is finite.

Open Questions. **1.** Let us call a u.b. space $\mathfrak{X}$ *flat* if there exists a quasi-transitive (option: and discrete) action of a group G on it by l.s. isomorphisms for which $\mathrm{S}(G) = G$.

What spaces are flat? It is known that a quasi-geodesic metric space is flat (for a discrete action) if and only if it is quasi-isometric to a finite dimensional Euclidean space $\mathbb{R}^n$.

What groups have flat Cayley spaces? Does the class of the flat groups coincide with the class of FC-groups?

2. Let $\mathfrak{X}$ be a u.b. space. Let $O(\mathfrak{X})$ be the minimal pseudoequivalence for which any group acting on $\mathfrak{X}$ by l.s. isomorphisms acts on $\mathfrak{X}$ by shifts respectively to the pseudoequivalence $O(\mathfrak{X})$.

Does $O(\mathfrak{X})$ coincide with $O(R)$ for the standard pseudoequivalence of a (quasi-geodesic) metric space? What spaces (groups) have virtually regular $O(\mathfrak{X})$-spaces (Cayley spaces)?

6.2 Pseudoequivalences Induced by Actions

6.2.1 Pseudoequivalences Induced on the Group

Let a group G be acting on a connected uniformly bounded space $\mathfrak{X}$.

Definition *A pseudoequivalence Γ_G of the group G induced by the action on the u.b. space* $\mathfrak{X}$ consists of relations $\mathcal{R}$, such that for some (and thus for any) point $x \in \mathfrak{X}$ the set of subsets $\{\{g_1(x), g_2(x)\} : d(g_1, g_2) \leq \mathcal{R}\}$ is uniformly bounded in the space $\mathfrak{X}$.

Proposition 6.1. *If the action of G on $\mathfrak{X}$ is quasi-transitive, then G with the induced pseudoequivalence is l.s. isomorphic to $\mathfrak{X}$.*

If the quasi-transitive action is discrete, then the Cayley space of the group G coincides with the pseudoequivalence induced by the action.

6.2.2 Pseudoequivalences Induced by a Shift Group

Let a (pseudo)group G be acting on a set $\mathfrak{X}$. Let $\Gamma_{(G,\mathfrak{X})}$ be a pseudoequivalence generated by the set of relations

$$\left\{\mathcal{R}_g \stackrel{\text{def}}{=} \{(x, g(x)) : x \in \mathfrak{X} (\in \alpha(g))\} \cup \mathcal{O}\right\}_{g \in G}.$$

$\Gamma_{(G,\mathfrak{X})}$ is the minimal pseudoequivalence of the set $\mathfrak{X}$ for which G acts on $\mathfrak{X}$ by shifts. For the right action of G on itself the pseudoequivalence $\Gamma_{(G,G)}$ coincides with the Cayley pseudoequivalence.

The set of all partial bijections g of $\mathfrak{X}$, shift equivalent to the identity on $\alpha(g)$, is a pseudogroup (see [CGH, HS]). This pseudogroup will be denoted $\mathrm{S}(\mathfrak{X})$ and called *the general shift pseudogroup* of the u.b. space $\mathfrak{X}$ (of the pseudoequivalence).

The theory of regular u.b. spaces and the theory of pseudogroups are equivalent. The following theorem explains this.

Theorem 6.2. *Let G be a pseudogroup acting on the set $\mathfrak{X}$. The general shift pseudogroup of the pseudoequivalence $\Gamma_{(G,\mathfrak{X})}$ coincides with G.*

If $\mathfrak{X}$ is a regular u.b. space, then its pseudoequivalence is equal to $\Gamma_{(\mathrm{S}(\mathfrak{X}),\mathfrak{X})}$.

Thus the theory of pseudogroups is essentially geometrical.

7 Measures on Uniformly Bounded Spaces

Here we are going to present the theory of invariant mean and amenability for u.b. spaces. Classical theory of invariant mean on (semi)groups has originated from works of Banach, Tarski, von Neumann (see [BT, Neu, Gre, Day]).

Recently, a theory of amenable metric spaces was developed (see [G, N3, CGH, W]). In fact, the classical Følner amenability criterion ([F]) for groups is geometrical and can be accepted as a definition of an amenable metric space.

The theory of invariant mean on u.b. spaces generalizes all these theories. It gives simple and natural proofs of the classical theorems and criteria.

Another general theory based on the notion of a pseudogroup is described in [CGH]. This theory is of the same generality, as follows from the theorem 6.2.

7.1 Uniformly Bounded Measures

In the current section $\mathfrak{X}$ is a u.b. space.

Definition. A (finitely additive) measure μ, defined on the bounded sets of the space $\mathfrak{X}$, is *uniformly bounded* if for any $\mathcal{R} \in \Gamma(\mathfrak{X})$ we have

$$\sup_{x \in \mathfrak{X}} \mu(B_x(\mathcal{R})) < \infty.$$

We denote the set of all u.b. measures by a symbol $l_\infty^+(\mathfrak{X})$. This set bears a natural structure of a commutative semigroup respectively to the addition. We can also multiply the elements of $l_\infty^+(\mathfrak{X})$ on real positive constants.

The measure $\nu(A) \stackrel{\text{def}}{=} |A|$ is uniformly bounded (by $|A|$ we denote the cardinality of the set A) iff the space $\mathfrak{X}$ is regular.

If the space is regular, then any u.b. measure μ is uniquely defined by the function $f_\mu : \mathfrak{X} \to \mathbb{R}^+, f_\mu(x) = \mu(\{x\})$ (its *distribution*). Then the semigroup $l_\infty^+(\mathfrak{X})$ is isomorphic to the semigroup of all bounded functions $f : \mathfrak{X} \to \mathbb{R}^+$.

If the space is regular, then for any subset $A \subseteq \mathfrak{X}$ *the indicator* 1_A belongs to $l_\infty^+(\mathfrak{X})$. The indicator is defined by the equality $1_A(B) = |A \cap B|$.

7.2 Shift Equivalent Measures and Sets

Two sets $A, B \subset \mathfrak{X}$ are *shift equivalent* if there exists a bijection $f : A \to B$, which is a shift.

Let us define a notion of shift equivalent measures.

We say that $\mu_1 \in l_\infty^+(\mathfrak{X})$ *covers* $\mu_2 \in l_\infty^+(\mathfrak{X})$ (we use a notation $\mu_2 \sqsubseteq \mu_1$) if there exists $\mathcal{R}$ such that for all bounded $A \subset \mathfrak{X}$

$$\mu_1(\mathrm{Neib}(A, \mathcal{R})) \geq \mu_2(A).$$

The relation " $\sqsubseteq$ " is a preordering relation.

We say that $\mu_1, \mu_2 \in l_\infty^+(\mathfrak{X})$ are *shift equivalent* ($\mu_1 \approx \mu_2$), if $\mu_1 \sqsubseteq \mu_2$ and $\mu_2 \sqsubseteq \mu_1$.

We accept this definition as it seems to be technically convenient and agrees with the notion of a shift equivalent sets, since

Theorem 7.1. *Any two indicators of sets are shift equivalent iff the sets are shift equivalent.*

Proof. The "only if" part is trivial.

Suppose the indicators of the sets A_1, A_2 are shift equivalent. Then, for an appropriate $\mathcal{R}$, for any finite set $B \subseteq A_1$ we have:

$$|\mathrm{Neib}(B, \mathcal{R}) \cap A_2| \geq |B|.$$

Thus, by the P.Hall marriage theorem (see [Mir]), for any finite $B \subseteq A_1$ there exists an injection $F_B : B \to A_2$ such that $d(x, F_B(x)) \leq \mathcal{R}$.

Thus, there exists an injection $F_1 : A_1 \to A_2$ with the same property.

By the same argument there exists an injection $F_2 : A_2 \to A_1$ such that $d(x, F_2(x)) \leq \mathcal{R}$.

Now using the construction from the classical proof of the Cantor-Berstein theorem, we see that there exists a bijection $F : A_1 \to A_2$ such that $F(x) \in \{F_1(x), F_2^{-1}(x)\}$ for any $x \in A_1$. Thus A_1 and A_2 are shift equivalent. $\square$

If G is a group, then its two subsets $A_1, A_2 \subseteq G$ are shift equivalent if and only if there exist a partition of the set A_1 into disjoint subsets $A_1 = B_1 \uplus \cdots \uplus B_n$ and elements of the group $g_1, \ldots, g_n$, such that $A_2 = B_1 g_1 \uplus \cdots \uplus B_n g_n$ is a partition of the set A_2 into disjoint subsets.

If the group G can be presented as a union of two shift equivalent to G subsets, then we get the well known Banach-Tarski paradoxical partition for groups (see [Tar, Wag, HS]). The classical paradox about partitions of a sphere ([BT]) also can be formulated in the terms of shift equivalent sets in an appropriate u.b. space.

A metrical counterpart of paradoxical partitions is investigated in [DSS] and the general situation for pseudogroups is investigated in [CGH].

The relation " $\approx$ " is a congruence on the semigroup $l_\infty^+(\mathfrak{X})$, thus we may introduce a semigroup $\mathcal{L}_\infty^+(\mathfrak{X}) = l_\infty^+(\mathfrak{X})/\approx$. $\mathbb{R}^+$ also acts on $\mathcal{L}_\infty^+(\mathfrak{X})$ by the multiplication. $\mathcal{L}_\infty^+(\mathfrak{X})$ is ordered by the relation " $\sqsubseteq$ " (or more precisely by the induced relation).

One can prove that l.s. isomorphic spaces have isomorphic semigroups $\mathcal{L}_\infty^+(\mathfrak{X})$ (on the contrary, $l_\infty^+(\mathfrak{X})$ is not l.s. invariant).

It is interesting to investigate the connection between the semigroup $\mathcal{L}_\infty^+(\mathfrak{X})$ and other l.s. invariants of $\mathfrak{X}$. The most interesting cases at the moment are the metric spaces (groups) and the well known quasi-isometric invariants: growth, hyperbolicity, isoperimetric inequalities. The relation between the semigroup $\mathcal{L}_\infty^+(\mathfrak{X})$ and amenanability will be shown in this paper.

7.3 Invariant Mean and Amenable Spaces

Definition. *Invariant mean* on the space $\mathfrak{X}$ is a nontrivial semigroup homomorphism

$$m : l_\infty^+(\mathfrak{X}) \to \mathbb{R}^+$$

having equal values on shift equivalent measures. Or, equivalently, it is just a nontrivial homomorphism $m : \mathcal{L}_\infty^+(\mathfrak{X}) \to \mathbb{R}^+$.

This definition is a direct generalization of the classical definition from the group theory (see [Gre, CGH]).

A u.b. space is *amenable* if there exists an invariant mean on it.

Let us make some auxiliary constructions.

Definition. We say that $\mu_1, \mu_2 \in \mathcal{L}_\infty^+(\mathfrak{X})$ are *linearly equivalent* if there exists $\nu \in \mathcal{L}_\infty^+(\mathfrak{X})$ such that $\mu_1 + \nu = \mu_2 + \nu$. If μ_1 and μ_2 are linearly equivalent, then we write $\mu_1 \sim \mu_2$.

Obviously, the semigroup $\mathcal{L}_\infty^+(\mathfrak{X})/\sim$ is well defined and cancellative. Let $\mathbb{L}(\mathfrak{X})$ be the group of quotients of the semigroup $\mathcal{L}_\infty^+(\mathfrak{X})/\sim$.

Any invariant mean naturally induces a nontrivial homomorphism of the groups

$$m' : \mathbb{L}(\mathfrak{X}) \to \mathbb{R}.$$

We are going to prove the following (see theorem 7.6).

A virtually regular u.b. space $\mathfrak{X}$ is amenable iff $\mathbb{L}(\mathfrak{X})$ is nontrivial.

In order to prove this, we have to introduce a notion of a regular measure (a generalization of the notion of a regular net.)

7.4 Regular Measures

A measure $\mu \in l_\infty^+(\mathfrak{X})$ is called *($\mathcal{R}$)-regular* if

$$\inf_{x \in \mathfrak{X}} \mu(B_x(\mathcal{R})) > 0.$$

A set $A \subseteq \mathfrak{X}$ has a regular indicator iff it is a regular net.

The regular measures have a fine characterization in the semigroup $\mathcal{L}_\infty^+(\mathfrak{X})$.

Theorem 7.2. *A uniformly bounded space has a regular measure iff it is virtually regular.*

$\mu \in \mathcal{L}_\infty^+(\mathfrak{X})$ is an image of a regular measure iff for any $\nu \in \mathcal{L}_\infty^+(\mathfrak{X})$ there exist a constant $K > 0$ and an element $\phi \in \mathcal{L}_\infty^+(\mathfrak{X})$, such that $\nu + \phi = K\mu$, and thus $\nu \sqsubseteq K\mu$.

Proof. Concentrate μ and ν on an $\mathcal{R}^2$-separated net N of $\mathfrak{X}$, and put

$$K = \sup_{x \in N} \nu'(\{x\}) \left(\inf_{x \in N} \mu'(\{x\}) \right)^{-1},$$

where $\mu \approx \mu', \nu \approx \nu'$ and $\inf_{x \in N} \mu'(\{x\}) \neq 0$, but $\mu'(\{x\}) = \nu'(\{x\}) = 0$ for all $x \notin N$. ☐

7.5 Følner Criterion

Proposition 7.3. *Let μ be a u.b. measure. The following statements are equivalent:*

1. *$C\mu \sqsubseteq \mu$ for some $C > 1$;*

2. *$C\mu \sqsubseteq \mu$ for all $C > 1$;*

3. *$C\mu \approx \mu$ for some $C > 1$;*

4. *$C\mu \approx \mu$ for all $C > 1$.*

We say that a regular measure $\mu \in \mathcal{L}^+_\infty(\mathfrak{X})$ satisfies *the Følner condition* if it satisfies one of the statements of 7.3. A net is said to be *a Følner net* if its indicator satisfies the Følner condition. (Compare with the classical Følner criterion in [F, Nam, CGH], and with the criterion for a graph to be paradoxical in [DSS].)

In the terms of the semigroup $\mathcal{L}^+_\infty(\mathfrak{X})$ the notion "Følner element" coincides with the notion "idempotent". The set of all idempotents $\mathfrak{T}(\mathfrak{X})$ is a subsemigroup of $\mathcal{L}^+_\infty(\mathfrak{X})$. We call this commutative semigroup of idempotents *a semigroup of trees.* The semigroup of trees is l.s. invariant. It is easy to prove, that $\mathfrak{T}(\mathfrak{X})$ is trivial for metric spaces of subexponential growth. Is it true, that for any metric space (or group) of exponential growth, the semigroup $\mathfrak{T}(\mathfrak{X})$ is nontrivial?

The following theorems become simple exercises in semigroup theory after using the theorem 7.2.

Proposition 7.4. *A virtually regular space $\mathfrak{X}$ satisfies the Følner condition iff any its two regular measures are shift equivalent.*

Theorem 7.5. *If the group $\mathbb{L}(\mathfrak{X})$ is trivial, then any regular measure on $\mathfrak{X}$ satisfies the Følner condition.*

If some regular measure satisfies the Følner condition, then the group $\mathbb{L}(\mathfrak{X})$ trivial.

If the group $\mathbb{L}(\mathfrak{X})$ is not trivial, then any regular measure on $\mathfrak{X}$ is mapped onto a nontrivial element of $\mathbb{L}(\mathfrak{X})$.

7.6 The Main Theorem

Theorem 7.6. *A virtually regular u.b. space $\mathfrak{X}$ is amenable iff $\mathbb{L}(\mathfrak{X})$ is nontrivial.*

This theorem is proved just using the Zohrn lemma.

Thus, from the proved above follow the theorems.

Theorem 7.7. *A virtually regular u.b. space is nonamenable iff some (and thus all) its nets satisfy the Følner condition.*

Theorem 7.8. *Any two regular nets of a nonamenable uniformly bounded space are shift equivalent.*

Two l.s. isomorphic regular nonamenable spaces are isomorphic.

7.7 Amenable Discrete Groups

Definition. A group G is said to be *amenable* (see [Gre, G])if there exists a finitely additive measure $\mu : 2^G \to [0, 1]$ such that

1. $\mu(G) = 1$

2. $\mu(Ag) = \mu(A)$ for all $A \subset G, g \in G$.

Such a measure μ on the group is called *an invariant measure on the group.*

Theorem 7.9. *A finitely generated group G is amenable iff it is amenable as a uniformly bounded Cayley space.*

Proof. If a group G has a nonamenable Cayley u.b. space, then it satisfies the Følner condition and thus the measures ν and 2ν are shift equivalent. (Where $\nu(A) = |A|$.) Thus there exists a paradoxical partition on it and so it is nonamenable as a group. On the other hand, if the space is amenable, then there exists an invariant mean on it. Any such invariant mean induces an invariant measure μ on G equal on a set $A \subseteq G$ to the value of the invariant mean on the indicator. $\square$

Open Question. One can define a natural u.b. space of a locally compact group as having the minimal left-invariant pseudoequivalence, for which the compact subsets are bounded.

Is the notion of amenability for this u.b. space equivalent to the classical notion of amenability for the locally compact groups?

From 7.9 follows

Corollary 7.10. *A discrete group is nonamenable iff any two its cofinite sets (nets) are shift equivalent.*

Two quasi-isometric nonamenable finitely generated groups are bi-Lipschitz equivalent.

From the theorem 7.7 follows, as a direct corollary, the classical Følner criterion (see [F]). In the same way the amenability criterion for cancellative semigroups ([Nam]) can be deduced from the theorem 7.7.

The shift equivalence of the measures ν and 2ν (see the proof of the theorem 7.9) is equivalent to the Tarski criterion (see [Tar, Wag]).

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