ITERATED MONODROMY GROUPS

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Contents

1. Introduction

Iterated monodromy groups are algebraic invariants of topological dynamical systems (e.g., rational functions acting on the Riemann sphere). They encode in a computationally efficient way combinatorial information about the dynamical systems. In hyperbolic (expanding) case the iterated monodromy group contains all essential information about the dynamical system. For instance, the Julia set of the system can be reconstructed from the iterated monodromy group).

Besides their applications to dynamical systems (see, for instance [BN06] and [Nek08b]) iterated monodromy groups are interesting from the point of view of group theory, as they often possess exotic properties. In some sense their complicated structure is parallel to the complicated structure of the associated fractal Julia sets. In some cases the relation with the dynamical systems can be used to understand algebraic properties of the iterated monodromy groups.

Even though the main application of the iterated monodromy groups is dynamics, their origins are in algebra (however, some previous works in holomorphic dynamics contained constructions directly related to the iterated monodromy groups, see [HOV95, LM97, Pil00]). They were defined in 2001 in connection with the following construction due to R. Pink. Let $F(x)$ be a rational function over $\mathbb{C}$. Consider its iterations $F^n(x)$ and let Ω_n be the field obtained by adjoining to the field of functions $\mathbb{C}(t)$ all solutions of the equation $F^n(x) = t$ in an algebraic closure of $\mathbb{C}(t)$. Then Ω_n is an increasing sequence of fields; denote $\Omega = \bigcup_{n\ge 1} \Omega_n$. How to compute the Galois group of the extension $\Omega/\mathbb{C}(t)$?

Note that the Galois group naturally acts on the sets L_n of solutions of the equation $F^n(x) = t$. If $x \in L_n$ is a solution of this equation, then $F(x)$ is a solution of the equation $F^{(n-1)}(x) = t$. Hence, the union of the sets L_n has a natural structure of the vertex set of a rooted tree: every vertex $x \in L_n$ is connected to the vertex $F(x) \in L_{n-1}$. The Galois group of $\Omega/\mathbb{C}(t)$ acts hence on this tree by automorphisms and the action is faithful. So, the problem of computation of the Galois group can be reformulated as the question of computation of the action of this group on the rooted tree.

It follows from classical facts (see, for example [For81, Theorem 8.12]) that the action of the Galois group $\mathrm{Aut}(\Omega/\mathbb{C}(t))$ on the nth level L_n of the tree coincides with the monodromy action of the fundamental group $\pi_1(\widehat{\mathbb{C}} \setminus B, t_0)$ on the fiber $F^{-n}(t_0)$ of the covering $F^n : \widehat{\mathbb{C}} \setminus F^{-n}(B) \longrightarrow \widehat{\mathbb{C}} \setminus B$, where B is the set of critical values of F^n and $t_0 \in \widehat{\mathbb{C}} \setminus B$ is a base-point. We get hence a sequence G_n of finite permutation groups. Their inverse limit is the Galois group $\mathrm{Aut}(\Omega/\mathbb{C}(t))$. The first example of an explicit computation of the groups G_n for a polynomial appears in the paper [Pil00] of K. Pilgrim.

Especially interesting is the case when $F(x)$ is a *post-critically finite* rational function, i.e., when the orbit of every critical point of F under iterations of F is finite. Then all the monodromy groups G_n are quotients of the fundamental group $\pi_1(\widehat{\mathbb{C}} \setminus P, t)$, where P is the union of the orbits of critical values of F. The natural epimorphisms from G_{n+1} to G_n agree with the epimorphisms $\pi_1(\widehat{\mathbb{C}} \setminus P, t) \longrightarrow G_n$. Hence, the inverse limit of the groups G_n contains a dense subgroup which is a quotient of the finitely generated fundamental group.

Since the early 80s an effective way of describing automorphisms of rooted trees was developed. It uses the language of Meely automata and wreath recursions. Groups generated by finite automata became important examples of groups with unusual properties (see [Sid98, GNS00, BGN03, BGŠ03, GŠ07]). All the techniques of this theory could be readily applied to the question of R. Pink. A simple recursive formula for the action of the generators of the Galois group $\mathrm{Aut}(\Omega/\mathbb{C}(t))$ on the tree was found. It was noted that for many examples the obtained automorphisms of the rooted tree are defined by finite automata. The Galois group is then the closure of a group generated by the states of a finite automaton. This dense subgroup of the Galois group (the image of the fundamental group of the punctured sphere) was more interesting from the point of view of groups acting on rooted trees (and later from the point of view of dynamics) than the profinite Galois group. It is called now the *iterated monodromy group* $\mathrm{IMG}(F)$ of the function F. It is interesting that some of the iterated monodromy groups were defined before as interesting examples of groups generated by finite automata. For instance, $\mathrm{IMG}\left(z^2-1\right)$ was defined in [GŻ02a, GŻ02b] by R. Grigorchuk and A. Żuk as an example of a group generated by a three-state automaton which does not contain a free subgroup, but can not be constructed from groups of sub-exponential growth using group-theoretical operations, preserving amenability (passing to subgroups, quotients, direct limits and extensions). Later L. Bartholdi and B. Virag proved in [BV05] that this group is amenable. It is thus the first example of an amenable group which can not be constructed from groups of sub-exponential growth. Another example of a previously known iterated monodromy group is $\mathrm{IMG}\left(z^3(-3/2+i\sqrt{3}/2)+1\right)$ which appeared in [FG91] as an example of a group of intermediate growth (see also [BG00], where the spectrum of the action of this group on the boundary of the tree was computed).

More on relations between Galois theory and groups acting on rooted trees (in particular iterated monodromy groups) see the papers [AHM05, BJ07].

It was noted that in many cases the graphs of the action of the group generated by a finite automaton on the levels of the tree seem to converge to some limit space (see, for instance [BG00]). This observation was formalized later by the author of these notes in a notion of the *limit space of a contracting group* generated by an automaton. This theory was also inspired by the theory of numeration systems on $\mathbb{R}^n$ (see [Pen65, Knu69, Vin00]) and results of the paper [NS04].

The theory of limit spaces of groups generated by automata (of *self-similar groups*) was developed just in time to apply it to the iterated monodromy groups. It was shown that the limit space of the iterated monodromy group $\mathrm{IMG}(F)$ of a post-critically finite rational function is homeomorphic to the Julia set of F. This way a direct connection of the theory of groups generated by automata to holomorphic dynamics was established. Now all the exotic examples of groups generated by automata became a part of a theory with many connections with other branches of Mathematics.

The present paper is a survey of the theory of iterated monodromy groups, with emphasis on examples, applications and algebraic properties of groups. More details and proofs can be found in the book [Nek05] and in the articles [Nek08c, Nek09, Nek08a]. See also the surveys [BGN03, Nek07c, GŠ07].

The second chapter introduces the main constructions and some simple examples. In particular, we discuss the formula for computation of the iterated monodromy group.

Chapter 3 describes the main algebraic tools of the theory and lists some open questions on algebraic properties of the iterated monodromy groups.

Chapter 4 develops the theory of limit spaces of self-similar groups. We also show how iterated monodromy groups can be used to construct models of the Julia sets of dynamical systems.

Examples of interesting iterated monodromy groups and their applications are collected in the last chapter of the paper. This includes: examples of Abelian

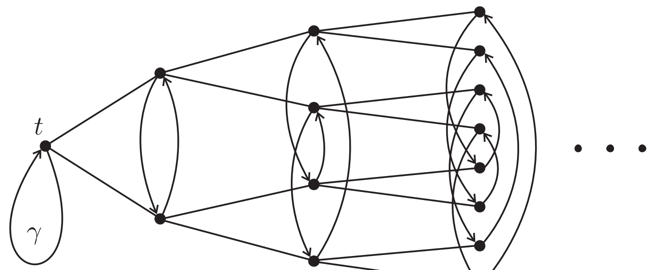

Figure 1. Monodromy action

iterated monodromy groups and their relation to self-affine tillings; virtually nilpotent iterated monodromy groups and a theorem of M. Shub; the Grigorchuk group and a family of iterated monodromy groups of the tent map, originally defined by Z. Šunić; iterated monodromy groups of quadratic polynomials and the Mandelbrot set; an example of the iterated monodromy group of an endomorphism of $\mathbb{CP}^2$; and an example of a group of non-uniform exponential growth.

I would like to thank the organizers of "Groups St Andrews 2009" in Bath for a beautiful conference and for inviting me to give the lectures, which are the basis of these notes.

2. Definitions and examples

2.1. **Definition.** Let $\mathcal{M}$ be a path connected and locally path connected topological space, and let $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a degree $d > 1$ covering map, where $\mathcal{M}_1 \subseteq \mathcal{M}$ is a subset of $\mathcal{M}$. Here a degree d covering map is a continuous map such that for every point $x \in \mathcal{M}$ there exists a neighbourhood $U \ni x$ and a decomposition of $p^{-1}(U)$ into a union of d disjoint subsets $U_1, \ldots, U_d$ such that $p : U_i \longrightarrow U$ is a homeomorphism.

We call such maps $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ *partial self-coverings* of $\mathcal{M}$. A partial self-covering can be iterated, as any partial map, and the iterates $f^n : \mathcal{M}_n \longrightarrow \mathcal{M}$ will be also partial self-coverings. Note that the domains $\mathcal{M}_n$ in general become smaller as n grows.

Choose a point $t \in \mathcal{M}$ and consider the fundamental group $\pi_1(\mathcal{M}, t)$ and the rooted *tree of preimages* with the vertex set

$$T_p = \bigsqcup_{n \geq 0} p^{-n}(t), \tag{1}$$

where a vertex $z \in p^{-(n+1)}(t)$ is connected by an edge with the vertex $p(z) \in p^{-n}(t)$. The point t is the root of the tree T_p.

The fundamental group $\pi_1(\mathcal{M}, t)$ acts on the levels $p^{-n}(t)$ of the tree T_p by the usual *monodromy action* (see, for instance, [Bre93, Section III.5]). The image of a point $z \in p^{-n}(t)$ under the action of a loop $\gamma \in \pi_1(\mathcal{M}, t)$ is the endpoint of the unique lift of γ by p^n starting at z (see Figure 1).

Since the action is defined by lifting the loops by p, the monodromy action of the fundamental group on the levels of the tree agrees with the tree structure, and we get an action of $\pi_1(\mathcal{M}, t)$ on T_p by automorphisms of the rooted tree. This action is called the *iterated monodromy action*.

The iterated monodromy action is not faithful in general, i.e., there exist loops $\gamma \in \pi_1(\mathcal{M}, t)$ which are lifted by iterations of p only to loops. The quotient of the fundamental group by the kernel of the iterated monodromy action (i.e., by the subgroup of loops lifted only to loops) is called the *iterated monodromy group of* p and is denoted IMG (p).

2.2. **Example: double self-covering of the circle.** Consider the orientation-preserving degree two self-covering of the circle. We realize it as the self-map $p : x \mapsto 2x$ of the circle $\mathbb{R}/\mathbb{Z}$.

The fundamental group of the circle is generated by the loop γ equal to the image of $[0, 1]$ in $\mathbb{R}/\mathbb{Z}$. The lifts of γ by the iterations p^n are obviously the images of the segments $\left[\frac{m}{2^n}, \frac{m+1}{2^n}\right]$, for $m = 0, \ldots, 2^n - 1$, in $\mathbb{R}/\mathbb{Z}$. We get hence a cycle of 2^n arcs.

It follows that the generator γ of the fundamental group acts as a transitive cycle on every level of the tree of preimages. Up to conjugacy in the automorphism group of the tree, there is only one such an automorphism (see [BORT96, GNS01]). It is called the *adding machine*.

We conclude that the iterated monodromy group of the double self-covering of the circle is isomorphic to the fundamental group $\mathbb{Z}$ of the circle. More interesting examples will follow.

2.3. **Chebyshev polynomials.** Chebyshev polynomials can be defined by the equality

$$T_d(x) = \cos(d \arccos x) = \frac{1}{2}\left(\left(x + \sqrt{x^2 - 1}\right)^d + \left(x - \sqrt{x^2 - 1}\right)^d\right)$$

or by an equivalent recursive formula

$$T_{d+1}(x) = 2xT_d(x) - T_{d-1}(x) \tag{2}$$

with the initial values $T_0(x) = 1$ and $T_1(x) = x$. The first Chebyshev polynomials are

$$T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1.$$

The polynomials T_d (divided by the leading coefficient 2^{n-1}) were defined by P. Chebyshev in [Che54] in relation with problems of approximation theory. They were of course known much earlier. See, for instance, Section 243 in Chapter XIV of L. Euler's *"Introductio in Analysin Infinitorum"* [Eul48, Eul88], where the polynomials up to T_7 together with a general formula for T_d are given.

It follows directly from the definition that T_d composed with T_k coincides with T_{dk}. In particular, nth iteration of T_d is T_{d^n}.

Differentiating the equality $T_d(\cos\theta) = \cos d\theta$ we get $T'_d(\cos\theta)\sin\theta = d \sin d\theta$, hence

$$T'_d(\cos\theta) = \frac{d \sin d\theta}{\sin\theta}.$$

It follows that the critical points of T_d are $\cos\frac{\pi m}{d}$ for $m = 1, 2, \ldots, d - 1$.

The set of values of T_d at the critical points is $\{\cos \pi m \ : \ m = 1, 2 \ldots, d - 1\}$, which is equal to $\{1, -1\}$ for $d \geq 3$ and to $\{-1\}$ for $d = 2$. We have

$$T_d(1) = 1, \qquad T_d(-1) = (-1)^d.$$

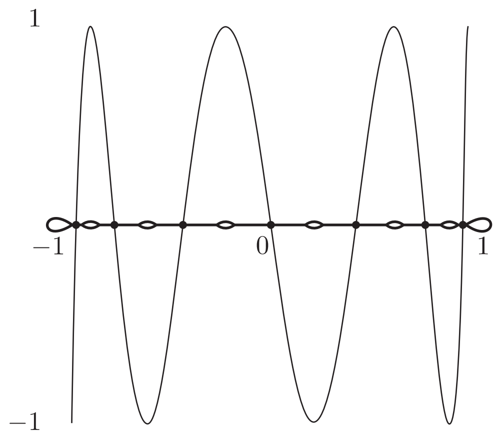

FIGURE 2. Computing IMG (T_d)

It follows that the set $\{-1, 1\}$ is T_d-invariant, hence the Chebyshev polynomial T_d is a partial self-covering

$$T_d : \mathbb{C} \setminus T_d^{-1}(\{-1, 1\}) \longrightarrow \mathbb{C} \setminus \{-1, 1\}.$$

Let us describe the iterated monodromy action of $\pi_1(\mathbb{C} \setminus \{\pm 1\})$ associated with the polynomial T_d. Choose $t = 0$ as a base-point of the fundamental group. Denote by a a small loop around 1 attached to 0 by a straight segment. Similarly, let b be a small loop around -1 connected to 0 by a straight segment. These two loops generate $\pi_1(\mathbb{C} \setminus \{\pm 1\}, t)$. Let us find their lifts by the nth iterate T_{d^n} of the polynomial T_d. We have

$$T_{d^n}^{-1}(0) = \left\{ \cos \frac{\pi/2 + l\pi}{d^n} \;:\; l = 0, \ldots, d^n - 1 \right\},$$

i.e., $T_{d^n}^{-1}(0)$ is the set of points obtained by projecting onto the x-axis the vertices of the regular $2d^n$-gon P_n inscribed into the unit circle in such a way that the x-axis is its non-diagonal axis of symmetry. The critical values of T_{d^n} are the projections of the midpoints of the arcs connecting consecutive vertices of P_n. Equality $T_{d^n}(\cos \theta) = \cos d^n \theta$ implies now that the lifts of a and b are obtained by projecting the arcs connecting consecutive vertices of P_n. We get in this way a path of edges with loops at the ends connecting d^n vertices. See Figure 2.3, where the graph of T_7 together with the graph of the action of $\langle a, b \rangle$ on $T_7^{-1}(0)$ is shown.

It follows that the elements of IMG (T_d) corresponding to the generators a and b are involutions and that their product is infinite, since it is transitive on each level of the tree. Consequently, the group IMG (T_d) is infinite dihedral.

2.4. **Computation.** The tree of preimages (1) is an abstract rooted tree, so if we want to compute the iterated monodromy action of the fundamental group on it, we need to introduce some "coordinates" on the tree.

Vertices of a regular rooted trees are often encoded by finite words over an alphabet X. The root is the empty word $\emptyset$. A vertex represented by a word v is connected to the vertices of the form vx for $x \in \mathsf{X}$. Denote by X^* the set of all finite words over the alphabet X seen as a rooted tree.

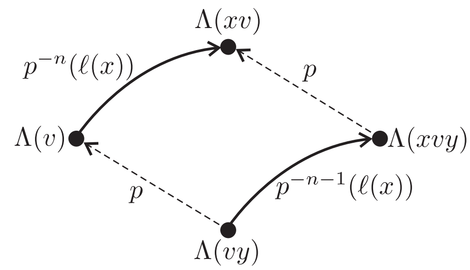

FIGURE 3. The map Λ

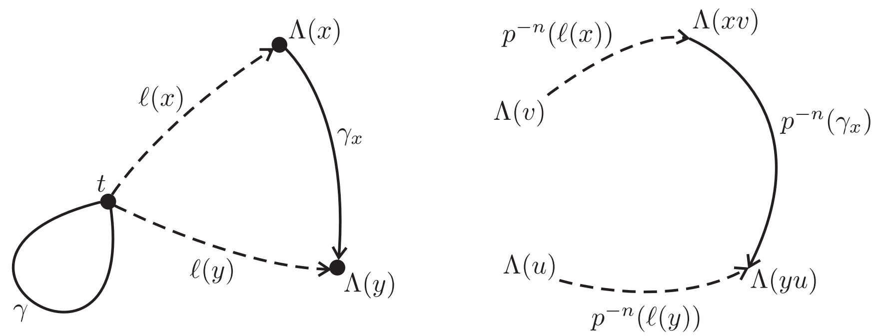

FIGURE 4. Recurrent formula and its proof

There exists a convenient encoding of the vertices of the tree of preimages T_p by words, which uses lifts of paths. Let the size of the alphabet X be equal to the degree of the partial self-covering $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$. Choose a bijection $\Lambda : \mathsf{X} \longrightarrow p^{-1}(t)$ and a path $\ell(x)$ from t to $\Lambda(x)$ for every $x \in \mathsf{X}$.

We also set $\Lambda(\emptyset) = t$ and define the map $\Lambda : \mathsf{X}^* \longrightarrow T_p$ inductively by the rule:

$$\Lambda(xv) \text{ is the end of the } p^{|v|}\text{-lift of } \ell(x) \text{ starting at } \Lambda(v).$$

It is easy to prove by induction (see Figure 3) that the defined map $\Lambda : \mathsf{X}^* \longrightarrow T_p$ is an isomorphism of rooted trees.

Let us conjugate the iterated monodromy action on the tree T_p by the isomorphism Λ, i.e., let us identify the trees T_p and X^* using the isomorphism Λ. We get a *standard action* of the iterated monodromy group $\mathrm{IMG}\,(p)$ on the tree X^*. The standard action is computed in the following recursive way.

Proposition 2.1. *Let γ be an element of the fundamental group $\pi_1(\mathcal{M}, t)$. For $x \in \mathsf{X}$, let γ_x be the lift of γ by p starting at $\Lambda(x)$. Let $y \in \mathsf{X}$ be such that $\Lambda(y)$ is the end of γ_x. Then for every $v \in \mathsf{X}^*$ we have*

$$\gamma(xv) = y\left(\ell(x)\gamma_x\ell(y)^{-1}\right)(v).$$

The loop $\ell(x)\gamma_x\ell(y)^{-1}$ is shown on the left-hand side part of Figure 4. The proof of the proposition is shown on the right-hand side part. It shows a lift of the path $\ell(x)\gamma_x\ell(y)^{-1}$ by the covering p^n.

2.5. **Examples.** Let us show how Proposition 2.1 is used to compute the action of generators of iterated monodromy groups on trees.

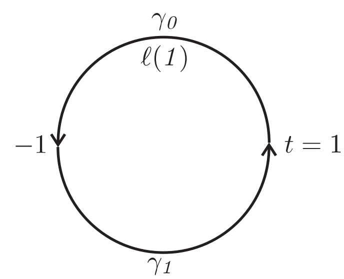

FIGURE 5. Computing IMG (z^2)

2.5.1. *The adding machine.* Consider the polynomial $p(z) = z^2$ as a map from the complex plane to itself. It induces a double self-covering of $\mathbb{C} \setminus \{0\}$ (homotopically equivalent to the 2-fold self-covering $x \mapsto 2x$ of the circle $\mathbb{R}/\mathbb{Z}$).

Choose the base-point $t = 1$. We have $p^{-1}(1) = \{1, -1\}$. Let us take the alphabet $\mathsf{X} = \{0, 1\}$. Let $\ell(0)$ be the trivial path at the base-point, and let $\ell(1)$ be the unit upper half-circle. Let γ be the unit circle based at t with the positive orientation.

We get, applying Proposition 2.1, the following recurrent formula for the action of γ on X^* (see Figure 5):

$$\gamma(0v) = 1v, \qquad \gamma(1v) = 0\gamma(v).$$

This transformation is known as the *adding machine*, or *odometer*, since it describes the process of adding one to a binary integer. If the first digit (the last digit in the usual encoding of binary numbers) is zero then we change it to one, if it is one, we change it to zero and carry. Formally:

$$\gamma(x_0x_1 \ldots x_n) = y_0y_1 \ldots y_n \text{ if and only if } 1 + \sum_{k=1}^{n} x_k 2^k = \sum_{k=1}^{n} y_k 2^k \pmod{2^{n+1}}.$$

It follows that the action of γ is transitive on every level of the tree, which we already knew (see Subsection 2.2).

2.5.2. *Chebyshev polynomials.* Let us compute the standard action of the iterated monodromy group of the Chebyshev polynomial T_d. We choose $t = 0$ as the base-point. Connect it to the preimages $z \in \left\{\cos \frac{\pi/2 + l\pi}{d} \; : \; l = 0, \ldots, d-1\right\}$ by straight segments. Choose the alphabet $\mathsf{X} = \{0, 1, \ldots, d-1\}$ and the bijection

$$\Lambda : \mathsf{X} \longrightarrow T_d^{-1}(t) : l \mapsto \cos \frac{\pi/2 + l\pi}{d}.$$

Let a and b be small loops around the post-critical points 1 and -1, respectively, both connected to the base-point by straight segments. It follows from the description of the lifts of a and b given in Subsection 2.3 that the generators of a and b act by the rules

$$a(0v) = 0a(v), \quad a((d-1)v) = (d-1)b(v), \quad a(lv) = \alpha(l)v, \qquad \text{for } l = 1, 2, \ldots, d-2,$$
$$b(lv) = \beta(l)v, \qquad \text{for } l = 0, 1, \ldots, d-1,$$

for even d, where permutations α and β are

$$\alpha = (12)(34) \cdots (d-3, d-2), \quad \beta = (01)(12) \cdots (d-2, d-1).$$

If d is odd, then

$$a(0v) = 0a(v), \quad a(lv) = \alpha(l)v, \qquad \text{for } l = 1, 2 \dots, d-1,$$

$$b((d-1)v) = (d-1)b(v), \quad b(lv) = \beta(l)v, \qquad \text{for } i = 0, 1, \dots, d-2,$$

where

$$\alpha = (\mathit{12})(\mathit{34})\cdots(d-\mathit{2}, d-\mathit{1}), \quad \beta = (\mathit{01})(\mathit{23})\cdots(d-\mathit{3}, d-\mathit{2}).$$

In particular, the generators of the iterated monodromy group of the polynomial $T_2(z) = 2z^2 - 1$ are defined by the recursive rule

$$\begin{array}{rclrcl} a(\mathit{0}v) & = & \mathit{0}a(v), & a(\mathit{1}v) & = & \mathit{1}b(v), \\ b(\mathit{0}v) & = & \mathit{1}v, & b(\mathit{1}v) & = & \mathit{0}v. \end{array}$$

2.5.3. *The polynomial* $-\frac{z^3}{2} + \frac{3z}{2}$. A rational function $f(z) \in \mathbb{C}(z)$ is *post-critically finite* if the orbit (under iterations of f) of every critical point of f is finite. The union P_f of the orbits of the critical values of f is the *post-critical set* P_f of f. Simple examples of post-critically finite polynomials are z^n (with the post-critical set $\{0, \infty\}$) and Chebyshev polynomials T_n (with the post-critical set $\{1, -1, \infty\}$).

If f is post-critically finite, then it is a partial self-covering $f : \widehat{\mathbb{C}} \setminus f^{-1}(P_f) \longrightarrow \widehat{\mathbb{C}} \setminus P_f$, since $f^{-1}(P_f) \supset P_f$. Here $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.

Consider the polynomial $f(z) = -\frac{z^3}{2} + \frac{3z}{2}$. It has three critical points $\infty, 1, -1$, which are fixed under f. Hence it is post-critically finite and is a covering of $\mathbb{C}\setminus\{\pm 1\}$ by the subset $\mathbb{C} \setminus f^{-1}(\{\pm 1\}) = \mathbb{C} \setminus \{\pm 1, \pm 2\}$.

Choose the base-point $t = 0$. It has three preimages $0, \pm\sqrt{3}$. Take $\mathsf{X} = \{\mathit{0}, \mathit{1}, \mathit{2}\}$ and choose the connecting paths and generators a and b of $\pi_1(\mathbb{C} \setminus \{\pm 1\}, 0)$ as it is shown on the bottom part of Figure 6 (the path $\ell(\mathit{0})$ is trivial).

The generators a and b are lifted by f to the paths shown on the upper part of Figure 6. We get then the following recurrent description of the action of a and b on X^*.

$$\begin{array}{rclrclrcl} a(\mathit{0}v) & = & \mathit{1}v, & a(\mathit{1}v) & = & \mathit{0}a(v), & a(\mathit{2}v) & = & \mathit{2}v, \\ b(\mathit{0}v) & = & \mathit{2}v, & b(\mathit{1}v) & = & \mathit{1}v, & b(\mathit{2}v) & = & \mathit{0}b(v). \end{array}$$

We see that a and b act as binary adding machines on the sub-trees $\{\mathit{0}, \mathit{1}\}^*$ and $\{\mathit{0}, \mathit{2}\}^*$, respectively.

Let us show that in this case the iterated monodromy group is different from the fundamental group of the punctured plane (which is freely generated by a and b).

We have

$$\begin{array}{rclrclrcl} a^2(\mathit{0}v) & = & \mathit{0}a(v), & a^2(\mathit{1}v) & = & \mathit{1}a(v), & a^2(\mathit{2}v) & = & \mathit{2}v, \\ b^2(\mathit{0}v) & = & \mathit{0}b(v), & b^2(\mathit{1}v) & = & \mathit{1}v, & b^2(\mathit{2}v) & = & \mathit{2}b(v), \end{array}$$

hence

$$[a^2, b^2](\mathit{0}v) = \mathit{0}[a, b](v), \quad [a^2, b^2](\mathit{1}v) = \mathit{1}v, \quad [a^2, b^2](\mathit{2}v) = \mathit{2}v.$$

It follows now that $[a^2, b^2]$ and $[a^2, b^2]^a$ commute. In fact, we will prove later that iterated monodromy groups of post-critically finite rational functions do not contain free subgroups.

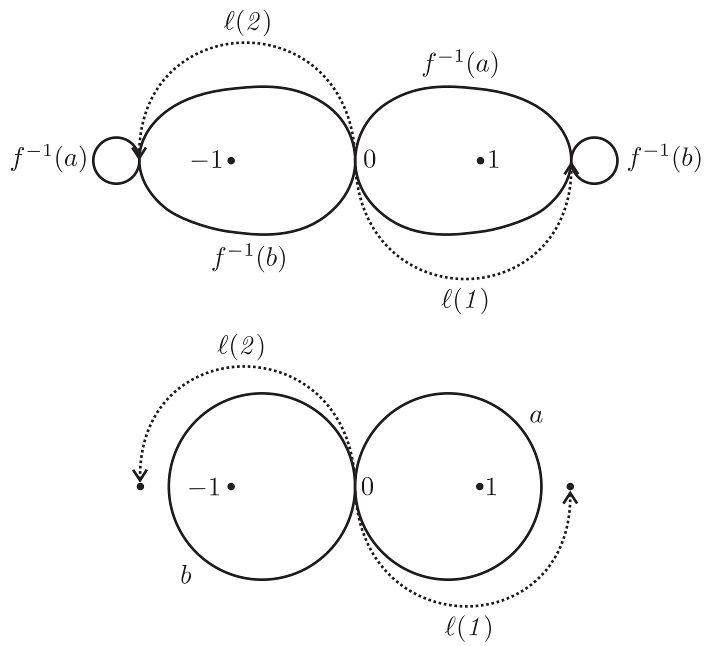

FIGURE 6. Computation of IMG $\left(-\frac{z^3}{2}+\frac{3z}{2}\right)$

2.5.4. *A two-dimensional example.* Consider the transformation $F(x,y)=\left(1-\frac{y^2}{x^2},1-\frac{1}{x^2}\right)$ of $\mathbb{C}^2$. It can be naturally extended to the projective plane $\mathbb{CP}^2$ by the formula

$$[x:y:z]\mapsto[x^2-y^2:x^2-z^2:x^2]$$

in homogeneous coordinates. The Jacobian of this map is

$$\left|\begin{array}{ccc} 2x & -2y & 0\\ 2x & 0 & -2z\\ 2x & 0 & 0\end{array}\right| = 8xyz,$$

hence the set of critical points of F is $\{x=0\}\cup\{y=0\}\cup\{z=0\}$. The orbits of the post-critical lines are:

$$\{x=0\}\mapsto\{z=0\}\mapsto\{y=z\}\mapsto\{x=y\}\mapsto\{x=0\}$$

and

$$\{y=0\}\mapsto\{x=z\}\mapsto\{y=0\}.$$

It follows that the post-critical set of F is (in the affine coordinates) the union of the line at infinity and the lines $x=0$, $x=1$, $y=0$, $y=1$, $x=y$.

The iterated monodromy group of F, as computed by J. Belk and S. Koch (see [BK08]), is generated by the transformations:

$$\begin{array}{rclrclrcl}
a(1v) &=& 1b(v), & b(1v) &=& 1c(v), & c(1v) &=& 4d(v),\\
a(2v) &=& 2v, & b(2v) &=& 2c(v), & c(2v) &=& 3(ceb)^{-1}(v),\\
a(3v) &=& 3v, & b(3v) &=& 3v, & c(3v) &=& 2(fa)^{-1}(v),\\
a(4v) &=& 4b(v), & b(4v) &=& 4v, & c(4v) &=& 1v,
\end{array}$$

$$\begin{array}{lllllllll}
d(1v) & = & 2v, & e(1v) & = & 1f(v), & f(1v) & = & 3b^{-1}(v), \\
d(2v) & = & 1a(v), & e(2v) & = & 2v, & f(2v) & = & 4v, \\
d(3v) & = & 4v, & e(3v) & = & 3f(v), & f(3v) & = & 1eb(v), \\
d(4v) & = & 3a(v), & e(4v) & = & 4v, & f(4v) & = & 2e(v).
\end{array}$$

3. Self-similar groups and virtual endomorphisms

3.1. **Self-similar groups.** We have seen in Proposition 2.1 that for every element g of the iterated monodromy group $\mathrm{IMG}\,(p)$ and for every $x \in \mathsf{X}$ there exists $g_x \in \mathrm{IMG}\,(p)$ such that

$$g(xv) = g(x)g_x(v)$$

for all $v \in \mathsf{X}^*$.

Definition 1. A group G acting faithfully on the set X^* is called *self-similar* if for every $g \in G$ and every $x \in \mathsf{X}$ there exist $h \in G$ such that

$$g(xw) = g(x)h(w)$$

for all $w \in \mathsf{X}^*$.

If the action of G on X^* is self-similar, then for every $v \in \mathsf{X}^*$ and every $g \in G$ there exists $h \in G$ such that

$$g(vw) = g(v)h(w)$$

for all $w \in \mathsf{X}^*$. The element h is uniquely defined, is called *section* (or *restriction*) of g in v, and is denoted $g|_v$. We have the following obvious properties:

$$g|_{v_1v_2} = g|_{v_1}|_{v_2}, \qquad (g_1g_2)|_v = g_1|_{g_2(v)}g_2|_v \tag{3}$$

for all $v, v_1, v_2 \in \mathsf{X}^*$ and $g, g_1, g_2 \in G$.

Let us take $\mathsf{X} = \{1, 2, \ldots, d\}$. For every $g \in G$ consider the element

$$\pi(g|_1, g|_2, \ldots, g|_d) \in S_d \ltimes G^d = S_d \wr G,$$

where $\pi \in S_d$ is the action of g on the set of words of length one (i.e., on the first level of the tree X^*). It is easy to check that the map

$$g \mapsto \pi(g|_1, g|_2, \ldots, g|_d),$$

is a homomorphism from G to $S_d \wr G$ (use (3)). This homomorphism is called the *wreath recursion* associated with the self-similar group G. In general a wreath recursion on a group G is any homomorphism $\Phi : G \longrightarrow S_d \wr G$.

The wreath recursion associated with the standard action of $\mathrm{IMG}\,(p)$ depends on the choice of the bijection of X with $p^{-1}(t)$ and on the choice of the connecting paths $\ell(x)$. Different choices produce wreath recursions which differ from each other by an inner automorphism of $S_d \wr G$.

We say that $\Phi_1, \Phi_2 : G \longrightarrow S_d \wr G$ are *equivalent* if there exists an inner automorphism τ of $S_d \wr G$ such that $\Phi_2 = \tau \circ \Phi_1$.

Every wreath recursion defines an action on the tree $\{1, 2, \ldots, d\}^*$. If $\Phi(g) = \pi(g_1, g_2, \ldots, g_d)$ then we put

$$g(iv) = \pi(i)g_i(v)$$

for all $v \in \{1, 2, \ldots, d\}^*$ and $x \in \{1, 2, \ldots, d\}$. These recurrent rules uniquely define the *action of G associated with Φ*.

The *faithful self-similar group* defined by the wreath recursion Φ is the quotient of G by the kernel of the associated action. Equivalent wreath recursions define self-similar groups which are conjugate in the full automorphism group of the rooted tree X^*.

If G is generated by a finite set $\{g_1, g_2, \ldots, g_k\}$, then the wreath recursion is determined by its values on the generators:

$$\begin{aligned}\Phi(g_1) &= \pi_1(g_{11}, g_{12}, \ldots, g_{1d}),\\ \Phi(g_2) &= \pi_2(g_{21}, g_{22}, \ldots, g_{2d}),\\ &\vdots\\ \Phi(g_k) &= \pi_k(g_{k1}, g_{k2}, \ldots g_{kd}).\end{aligned}$$

If we write g_{ij} as groups words in $g_1, \ldots, g_k$, then we get a finite description of the associated self-similar group. (As a wreath recursion over the free group.) We will often omit Φ and write just $g = \pi(g_1, g_2, \ldots, g_d)$, identifying the automorphism group $\mathrm{Aut}(\mathsf{X}^*)$ of the tree X^* with the wreath product $S_d \wr \mathrm{Aut}(\mathsf{X}^*)$.

Let $\Phi : G \longrightarrow S_d \wr G$ be a wreath recursion. Denote by K_Φ the kernel of the associated action on the tree. If $g \notin K_\Phi$, then there exists a finite word $v \in \mathsf{X}^*$ moved by g. Hence there exists an algorithm which stops if and only if g is not trivial in the self-similar group defined by Φ.

It is not known if every finitely generated self-similar has solvable word problem. Nevertheless, in some cases there exists a simple algorithm solving the word problem. Let E_1 be the kernel of Φ. Denote

$$E_{n+1} = \Phi^{-1}(\{1\} \cdot E_n^d), \tag{4}$$

and $E_\infty = \bigcup_{n\ge 1} E_n$. If the word problem is solvable in G, then there is an algorithm which, given an element $g \in G$, stops if and only if $g \in E_\infty$. If, additionally, $E_\infty = K_\Phi$, then we get a solution of the word problem in G/K_Φ. We will define later a class of self-similar groups for which this approach works and produces a polynomial time algorithm.

3.2. **Virtual endomorphisms.**

Definition 2. A *virtual endomorphism* $\phi : G \dashrightarrow G$ of a group G is a homomorphism $\mathrm{Dom}\,\phi \longrightarrow G$ from a subgroup of finite index $\mathrm{Dom}\,\phi < G$ to G.

If ϕ_1, ϕ_2 are virtual endomorphisms of G, then their composition is also a virtual endomorphism. Its domain is $\mathrm{Dom}\,\phi_1 \circ \phi_2 = \phi_2^{-1}(\mathrm{Dom}\,\phi_1)$.

In particular, if ϕ is a virtual endomorphism of G, then the iterates ϕ^n are also virtual endomorphisms.

Let $\Phi : G \longrightarrow S_d \wr G$ be a wreath recursion. Suppose that the projection of $\Phi(G)$ onto S_d is transitive, i.e., that the group G acts transitively on the first level of the tree. Fix a letter $x \in \mathsf{X}$. The *associated virtual endomorphism* is the map $\phi : g \mapsto g|_x$ from the stabilizer of $x \in \mathsf{X}$ to G, i.e., it is the map defined by the condition

$$g(xw) = x\phi(g)(w)$$

for all $g \in \mathrm{Dom}\,\phi$ and $w \in \mathsf{X}^*$.

The virtual endomorphism uniquely determines the wreath recursion (up to inner automorphisms of the wreath product). Namely, if $\{r_1, r_2, \ldots, r_d\}$ is a left coset representative system for $\mathrm{Dom}\,\phi < G$, then we define

$$\Phi_1(g) = \pi(g_1, \ldots, g_d),$$

where $\pi(i) = j$ if and only if $gr_i \,\mathrm{Dom}\,\phi = r_j \,\mathrm{Dom}\,\phi$, and $g_i = \phi(r_j^{-1} g r_i)$. Then Φ_1 is equivalent to Φ.

Two virtual endomorphisms associated with a wreath recursion (and possibly different letters $x \in \mathsf{X}$) are *conjugate* to each other, i.e., can be obtained one from the other by pre- and post-composition with inner automorphisms of G.

If $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a partial self-covering, then $\pi_1(\mathcal{M}_1)$ is identified with a subgroup of finite index in $\pi_1(\mathcal{M})$ and the virtual endomorphism associated with the standard iterated monodromy action is the map $\pi_1(\mathcal{M}_1) \longrightarrow \pi_1(\mathcal{M})$ induced by the inclusion $\mathcal{M}_1 \hookrightarrow \mathcal{M}$ (defined up to inner automorphisms of $\pi_1(\mathcal{M})$).

As an example, let us compute the virtual endomorphisms associated with the standard actions of $\mathrm{IMG}\left(z^2\right)$ and $\mathrm{IMG}\left(-z^3/2 + 3z/2\right)$.

If ϕ is the virtual endomorphism of $\mathbb{Z}$ associated with the wreath recursion

$$\Phi(\gamma) = (\mathit{01})(1, \gamma),$$

associated with $\mathrm{IMG}\left(z^2\right)$, then $\phi(\gamma^2) = \gamma$, since the stabilizer of any letter $x \in \{\mathit{0}, \mathit{1}\}$ is generated by γ^2 and $\Phi(\gamma^2) = (\gamma, \gamma)$. Therefore, the virtual endomorphism associated with the binary adding machine is the partial map $n \mapsto n/2$ on $\mathbb{Z}$.

The virtual endomorphism associated with $\mathrm{IMG}\left(-z^3/2 + 3z/2\right)$, i.e., with the wreath recursion

$$\Phi(a) = (\mathit{01})(1, a, 1), \quad \Phi(b) = (\mathit{02})(1, 1, b),$$

is

$$\begin{array}{rclcrcl} a^2 & \mapsto & a, & \qquad & b^{-1}ab & \mapsto & 1, \\ b^2 & \mapsto & b, & & a^{-1}ba & \mapsto & 1. \end{array}$$

We have the following description of the kernel of the action defined by a wreath recursion, see [Nek05, Proposition 2.7.5].

Proposition 3.1. *If $\phi : G \dashrightarrow G$ is the virtual endomorphism associated with a wreath recursion Φ, then the kernel K_Φ of the associated self-similar action is*

$$K_\Phi = \bigcap_{g \in G, n \geq 1} g^{-1} \cdot \mathrm{Dom}\,\phi^n \cdot g.$$

3.3. **Contracting groups.** The sections $g|_v$, defined above for self-similar groups can be naturally defined for arbitrary wreath recursion $\Phi : G \longrightarrow S_d \wr G$. We define $g|_v$ for $g \in G$ and $v \in \mathsf{X}^*$ inductively by $g|_\varnothing = g$ and by the condition

$$\Phi(g|_v) = \pi(g|_{v1}, g|_{v2}, \ldots, g|_{vd}).$$

We will have then for the action of G on X^* defined by the wreath recursion:

$$g(vw) = g(v) g|_v(w),$$

for all $v, w \in \mathsf{X}^*$.

Definition 3. A wreath recursion on G is *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that

$$g|_v \in \mathcal{N}$$

for all words v of length at least n.

The notion of a contracting group can be naturally formulated in terms of the associated virtual endomorphism in the following way.

Theorem 3.2. *Let $\phi : G \dashrightarrow G$ be a virtual endomorphism of a finitely generated group. Denote by $l(g)$ the length of a group element g with respect to a fixed finite generating set of G. Then the number*

$$\rho = \limsup_{n\to\infty} \sqrt[n]{\limsup_{g\in \mathrm{Dom}\,\phi^n, l(g)\to\infty} \frac{l(\phi^n(g))}{l(g)}}$$

does not depend on the choice of the generating set, and it is less than one if and only if the associated wreath recursion is contracting.

Since the virtual endomorphism associated with the iterated monodromy group $\mathrm{IMG}\,(p)$ maps a loop γ to its lift by p, expanding maps will have contracting iterated monodromy group. More precisely, the following theorem is proved in [Nek05, Theorem 5.5.3], where also a more detailed definition of an expanding covering is given.

Theorem 3.3. *If the partial self-covering $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is expanding, then* $\mathrm{IMG}\,(p)$ *is a contracting self-similar group.*

In particular, the iterated monodromy groups of post-critically finite rational functions are contracting.

3.4. **Algebraic properties of contracting groups.** In some sense, the class of the iterated monodromy groups of expanding maps can be identified with the class of contracting groups, since there exists a converse construction, which produces for every contracting self-similar group G an expanding self-covering (of orbispaces) $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ such that G is the iterated monodromy group of s. This construction (called the *limit dynamical system*) will be described later.

Let us list some known algebraic properties of contracting groups.

Theorem 3.4. *The word problem in a contracting self-similar group is solvable in polynomial time.*

If ρ is the contraction coefficient of the associated virtual endomorphism, then for every $\epsilon > 0$ there is an algorithm solving the word problem in degree $\frac{\log(|\mathsf{X}|)}{-\log\rho} + \epsilon$ time.

The algorithm is similar to the observation made in Subsection 3.1. See its description in [Nek05, Proposition 2.13.10]. It is a generalization of the algorithm described by R. Grigorchuk in [Gri85].

Besides the fact that the word problem is solvable in contracting groups (and the trivial fact that contracting groups act faithfully on the rooted tree, and hence are residually finite), the only other known general fact about algebraic properties of all contracting groups is absence of free subgroups, proved in [Nek07a]. The proof is based on the following general theorem which can be also used in many other situations.

Theorem 3.5. *Let G be a group acting faithfully on a locally finite rooted tree T. Denote by ∂T the boundary of T. Then one of the following is true:*

1. *G has no free subgroups;*
2. *there is a free non-abelian subgroup $F \leq G$ and a point $\xi \in \partial T$ such that the stabilizer F_ξ is trivial;*

(3) *there is a point $\xi \in \partial T$ and a free non-abelian subgroup $F \leq G_\xi$ such that F acts faithfully on all neighbourhoods of ξ.*

Proof. Let us sketch the proof of this theorem. For more details, see [Nek07a]. Suppose that G is a counterexample. Fix any free non-abelian subgroup $F < G$. For every $\xi \in \partial T$ the stabilizer F_ξ will be also free non-abelian, and there will exist a neighbourhood U of ξ and a non-trivial element $g \in F$ acting trivially on U. We get an open covering of ∂T by such subsets U.

By compactness and by the basic properties of topology on ∂T, we can find a finite covering of ∂T by disjoint subsets $\{U_i\}_{i=1,\ldots,k}$ such that the pointwise stabilizer of U_i is non-trivial for every i. Let F_1 be the subgroup of the elements of F leaving the sets U_i invariant. It has finite index in F. It follows that the pointwise stabilizer in F_1 of each set U_i is also non-trivial. The intersection of these stabilizers has to be trivial, since the action of G is faithful and the sets U_i cover ∂T. But intersection of non-trivial normal subgroups of a free group is always non-trivial. Hence we get a contradiction. □

As a corollary of Theorem 3.5 we get.

Theorem 3.6. *Contracting groups have no free subgroups.*

Proof. The boundary of the tree X^* is naturally identified with the space X^ω of the right-infinite words over the alphabet X.

Third option of Theorem 3.5 is not possible, since the sections (i.e., restrictions onto the neighbourhoods of a point $\xi \in \mathsf{X}^\omega$) of elements of G eventually belong to a finite set.

Let us show why the second option is also impossible. Let $w \in \mathsf{X}^\omega$ be a point of the boundary. Consider the *growth function of the orbit* $G(w)$ defined as $\gamma_w(r) = |B_w(r)|$, where

$$B_w(r) = \{g(w)\ :\ g \in G, l(g) \leq r\}$$

is the ball of radius r in the orbit $G(w)$. Here $l(g)$ is the length of the element $g \in G$ with respect to a fixed generating set of G (we may assume that G is finitely generated, since every finitely generated subgroup of G is contained in a finitely generated self-similar contracting subgroup of G). Consider the map $S : \mathsf{X}^\omega \longrightarrow \mathsf{X}^\omega$ erasing the first n letters of an infinite word. It is a $|\mathsf{X}|^n$-to-one map. We have for every $w \in \mathsf{X}^\omega$

$$S(g(w)) = g|_v(S(w)),$$

where v is the beginning of length n of the word w. The action of the group G is contracting, hence we can choose n and C such that $l(g|_v) < \frac{1}{2}l(g)$ for all g such that $l(g) \geq C$. Then we have

$$S(B_w(r)) \subset B_{S(w)}(r/2) \cup B_w(C),$$

which implies that

$$\gamma_w(r) = |B_w(r)| \leq |\mathsf{X}|^n |B_{S(w)}(r/2)| + N,$$

where N is the number of elements of G of length less than C. Applying this inequality $\lfloor \log_2 r \rfloor$ times we get

$$\gamma_w(r) \leq N(1 + |\mathsf{X}|^n + |\mathsf{X}|^{2n} + \cdots + |\mathsf{X}|^{\lfloor \log_2 r \rfloor n}) < r^{(n+1)\log_2 |\mathsf{X}|} \frac{N}{|\mathsf{X}|^n - 1},$$

i.e., we get a polynomial estimate of the growth of the orbits of G. This implies that there is no free subgroup of G acting freely on the orbit of a point $w \in \mathsf{X}^\omega$. □

3.5. **Open questions.**

3.5.1. *Algorithmic problems.* It is not known if the conjugacy problem is solvable in contracting groups. For a solution of the conjugacy problem in the Grigorchuk group, which uses self-similarity, see the papers [Leo98, Roz98, LMU08].

Most of the other classical algorithmic problems for groups are open for contracting groups.

An interesting problem, with possible applications to dynamics, is the following algorithmic question.

Problem. Given a wreath recursion decide if it defines a contracting group.

3.5.2. *Amenability.* The following problem is one of the main problems in the subject of contracting groups (see, for instance [Gri05, Problem 3.3] and [BKN08, Nek07a]).

Problem. Are contracting groups amenable?

A group G is called *amenable* if there exists a finitely-additive measure μ defined on all subsets of G such that $\mu(G) = 1$ and $\mu(A \cdot g) = \mu(A)$ for all $A \subset G$ and $g \in G$. This notion was introduced by J. von Neumann [vN29] in relation with the Banach-Tarski paradox [BT24, Wag94] (amenable groups are precisely the groups which do not admit a Banach-Tarski paradox). The word "amenable" is due to M. Day [Day49]. For more on amenability, see [Gre69, Run02].

We have seen before that contracting groups have no free subgroups, which makes Problem 3.5.2 even more interesting.

The following is a corollary of a more general result on amenability of groups generated by "bounded automata", see [BKN08].

Theorem 3.7. *If f is a post-critically finite polynomial, then* $\mathrm{IMG}\,(f)$ *is amenable.*

The first non-trivial partial case of this theorem ($\mathrm{IMG}\left(z^2 - 1\right)$) was shown by L. Bartholdi and B. Virag [BV05]. This group is the first example of an amenable group which can not be constructed from groups of sub-exponential growth (which are all amenable) by the group-theoretical operations preserving amenability: extensions, direct limits, direct products and passing to a subgroup and a quotient.

3.5.3. *Presentations.*

Problem. Which contracting groups are finitely presented?

There are examples of contracting virtual nilpotent groups, see [Nek05, Section 6.1] and Subsection 5.2 of our paper. But all the other known examples of contracting groups are not finitely presented. Namely, in all the other known examples of contracting groups G there exists a finitely presented group $\widetilde{G}$ and a wreath recursion $\Phi : \widetilde{G} \longrightarrow S_d \wr \widetilde{G}$ such that G is the quotient of $\widetilde{G}$ by the kernel K_Φ of the associated self-similar action of $\widetilde{G}$; the sequences of subgroups E_n, defined by (4) is strictly increasing; and $K_\Phi = E_\infty$.

In many cases, contracting groups have *finite L-presentations (finite endomorphic presentation).* A finite L-presentation of a group G is given by a finite set of relations R and an endomorphism σ (or perhaps a finite collection of endomorphisms) of the free group such that the set $\bigcup_{n \geq 0} \sigma^n(R)$ is a set of defining relations of the group G. Different variations of this definition are possible (see [Gri98, Bar03a]). The following problem is still open.

Problem. Do all contracting groups have finite L-presentations? Is there any relation between the topology of a partial self-covering and L-presentations of its iterated monodromy group?

3.5.4. *Growth.* Many iterated monodromy groups have word growth intermediate between polynomial and exponential (see Theorem 5.13 below and discussion after it). Moreover, the first example of a group of intermediate growth, the Grigorchuk group, is the iterated monodromy group of a partial orbispace self-covering (see Subsection 5.3). But so far all we know are some isolated examples, without a general theory.

Problem. Describe contracting groups of intermediate growth.

3.6. **Iterated monodromy group of a correspondence.** We have defined in Section 2.1 iterated monodromy groups of partial self-coverings, i.e., of a covering map $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ together with an embedding $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$.

There is no reason to restrict to the case when ι is an embedding. We did not use anywhere injectivity of ι. Therefore, the following structure is a natural setting for iterated monodromy groups.

Definition 4. A *topological automaton* (or *topological correspondence*) is a pair of maps $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$, where p is a finite degree covering and ι is a continuous map.

In general, one has to consider not only topological spaces $\mathcal{M}$ and $\mathcal{M}_1$, but *orbispaces*, i.e., topological spaces represented locally as quotients of the action of finite groups on topological spaces. More on orbispaces and related structures, see [BH99, Chapter III.$\mathcal{G}$] and [Nek05, Chapter 4]. Since in all our examples the orbispaces will be *developable*, we will instead consider proper actions of groups on topological spaces later (see Definition 9).

Topological automata (under different names) appeared in the works [Kat04, IS08]. Topological automata can be iterated, formally speaking, exactly in the same way as partial self-coverings. Set $\mathcal{M}_0 = \mathcal{M}$, $p_0 = p$, $\iota_0 = \iota$ and define inductively a space $\mathcal{M}_n$, a covering $p_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ and a continuous map $\iota_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ by the pullback diagram

$$\begin{array}{ccc} \mathcal{M}_{n+1} & \xrightarrow{\iota_n} & \mathcal{M}_n \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_{n-1}} \\ \mathcal{M}_n & \xrightarrow{\iota_{n-1}} & \mathcal{M}_{n-1} \end{array} \tag{5}$$

i.e., $p_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ is the *induced covering*, see [Ste51].

More explicitly, the space $\mathcal{M}_n$ (in the case when $\mathcal{M}$ is a topological space) is homeomorphic to the subspace

$$\{(z_0, z_1, \ldots, z_n) \; : \; \iota(z_{k+1}) = f(z_k)\} \subset \mathcal{M}^{n+1} \tag{6}$$

of "orbits" of length n. One should think of ι as of an approximation of the identity map.

The iterated monodromy action of $\pi_1(\mathcal{M}, t)$ is defined in the same way as for partial a self-covering. It is the monodromy action of the fundamental group on the fibers $(f_0 \circ f_1 \circ \cdots \circ f_n)^{-1}(t)$ of the compositions of the coverings f_i.

An equivalent way of defining the iterated monodromy group of a topological automaton is to use virtual endomorphisms. The virtual endomorphism of $\pi_1(\mathcal{M})$

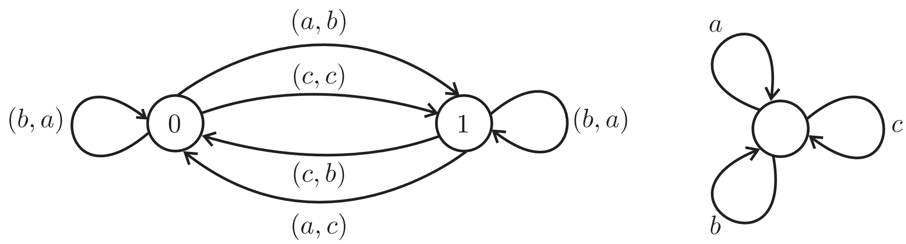

FIGURE 7. Dual Moore diagram

associated with the topological automaton is the homomorphism $\iota_* : \pi_1(\mathcal{M}_1) \longrightarrow \pi_1(\mathcal{M})$ induced by ι, where $\pi_1(\mathcal{M}_1)$ is identified with a subgroup of $\pi_1(\mathcal{M})$ by the isomorphism p_*. The self-similar group defined by the virtual endomorphism ι_* coincides with the iterated monodromy group of the topological automaton.

3.7. **Examples of topological automata.**

3.7.1. *Dual Moore diagrams.* Every virtual endomorphism ϕ of the free group can be realized as ι_* for a map $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$, where $\mathcal{M}$ is a bouquet of circles, and $\mathcal{M}_1$ is a finite covering graph of $\mathcal{M}$ defining the domain of ϕ.

Consequently, every self-similar group is an iterated monodromy group of a topological automaton over graphs. This correspondence can be constructed from the wreath recursion in the following way. Let G be a self-similar group acting on X^*. Let $\mathcal{M}$ be a bouquet of circles labelled by generators of G. Let $\mathcal{M}_1$ be the graph with the set of vertices $\mathsf{X} = \{1, 2, \ldots, d\}$, where for every generator g of G and every $x \in \mathsf{X}$ there is an arrow $e_{g,x}$ from x to $g(x)$. This arrow is mapped onto g by a map $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$, which is then obviously a covering. Define $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is such a way that it maps $e_{g,x}$ to the path corresponding to $g|_x$ (i.e., to any lift of $g|_x$ to the fundamental group of $\mathcal{M}$, which is the group freely generated by S).

It follows directly from the definitions that the iterated monodromy group of the obtained topological automaton $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ is isomorphic to G as a self-similar group.

Suppose that a generating set S of G has the property that for all $g \in S$ and $x \in \mathsf{X}$ the section $g|_x$ also belongs to S. Then S can be interpreted as the set of internal states of an *automaton* generating G. This automaton, taking a letter $x \in \mathsf{X}$ as input and being in a state $g \in S$, outputs the letter $g(x)$ and changes its internal state to $g|_x$. We can choose then in the automaton $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ the map ι to be cellular. Then the obtained graph $\mathcal{M}_1$ in which every arrow e is labelled by $(f(e), \iota(e))$ is called the *dual Moore diagram* of the automaton generating G.

Usual Moore diagram (also called *state diagram*) is the graph with the vertex set S in which for every $x \in \mathsf{X}$ and $g \in S$ we have an arrow from g to $g|_x$ labelled by $(x, g(x))$.

3.7.2. *Arithmetic-geometric mean of Lagrange and Gauss.* An example of a topological automaton originates from the *arithmetic-geometric mean*, studied by Gauss [Gau66] and Lagrange [Lag85]. On the history and applications of arithmetic-geometric mean see [Cox84, BB98, AB88].

It was shown by Lagrange in 1784 and independently by Gauss in 1791 that if a_0 and b_0 are positive real numbers, then the sequences

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}), \quad b_n = \sqrt{a_{n-1}b_{n-1}}$$

converge to a common value $M(a_0, b_0)$, called the arithmetic-geometric mean. One of its applications is the formula

$$\frac{\pi}{2M(a,b)} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}.$$

The arithmetic-geometric mean also gives very efficient algorithms of computing π and elementary functions, see [BB98].

In the complex case one has to choose one of two different values of the square root, so that we get a correspondence $(a, b) \mapsto ((a + b)/2, \sqrt{ab})$ rather than a map. This correspondence is homogeneous, hence

$$[z_1 : z_2] \mapsto [(z_1 + z_2)/2 : \sqrt{z_1 z_2}]$$

is a correspondence on the projective line $\widehat{\mathbb{C}}$. It is written in non-homogeneous coordinates as the correspondence $w \mapsto \frac{1+w}{2\sqrt{w}}$.

More formally, consider the following pair of maps on $\widehat{\mathbb{C}}$

$$f(w) = \frac{(1 + w)^2}{4w}, \quad \iota(w) = w^2.$$

Then an orbit of length n of the correspondence is a sequence $w_0, w_1, \ldots, w_n$ such that $w_{k+1} = \frac{1+w_k}{2\sqrt{w_k}}$, i.e., such a sequence that

$$\iota(w_{k+1}) = f(w_k).$$

Comparing it with (6), we see that iterating of the correspondence $[(z_1 + z_2)/2 : \sqrt{z_1 z_2}]$ is equivalent to iterating the topological automaton defined by the maps f and ι. The only remaining problem is that the map f is not a covering. Consider the set $\{0, 1, -1, \infty\} \subset \widehat{\mathbb{C}}$. The set of critical points of f is $\{-1, \infty\}$ and

$$f(\{0, 1, -1, \infty\}) = \{\infty, 1, 0\} = \iota(\{0, 1, -1, \infty\}).$$

Note that $f^{-1}(\{\infty, 1, 0\}) = \{\infty, 0, 1, -1\}$. It follows that if we denote $\mathcal{M} = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ and $\mathcal{M}_1 = \widehat{\mathbb{C}} \setminus \{0, 1, -1, \infty\}$, then $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a covering and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a continuous map. (Note that ι is also a covering map.) We get in this way a topological automaton $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ such that iterations of $\mathcal{F}$ correspond to iterations of the correspondence $[z_1 : z_2] \mapsto [(z_1 + z_2)/2 : \sqrt{z_1 z_2}]$.

Here we give a short summary (following [Cox84] and [Bul91]) of the properties of this automaton, which are essentially due to Gauss (see [Gau66] pp. 375–403).

Denote by $\mathfrak{H}$ the upper half plane $\{\tau \in \mathbb{C} : \Im(\tau) > 0\}$. Denote $z = e^{\pi i \tau}$ and let

$$p(\tau) = 1 + 2\sum_{n=1}^{\infty} z^{n^2}, \quad q(\tau) = 1 + 2\sum_{n=1}^{\infty} (-1)^n z^{n^2}.$$

Then $p(\tau)^2 + q(\tau)^2 = 2p(2\tau)^2$ and $p(\tau)q(\tau) = q(2\tau)^2$, i.e., $p(2\tau)^2$ is the arithmetic mean of $p(\tau)^2$ and $q(\tau)^2$, while $q(2\tau)^2$ is their geometric mean. If we denote $k(\tau) = q(\tau)^2/p(\tau)^2$, then our correspondence maps $k(\tau)$ to $k(2\tau)$, i.e., we have

$$f(k(\tau)) = k(2\tau)^2 = \iota(k(2\tau)). \tag{7}$$

Denote by $\Gamma(2)$ the subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ consisting of matrices

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \equiv \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \pmod 2.$$

It is freely generated by $\left[\begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array}\right]$ and $\left[\begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array}\right]$ (see [San47] and [Har00, II.B.25]). Denote also by $\Gamma_2(4)$ the index two subgroup of $\Gamma(2)$ consisting of matrices with $b \equiv 0 \pmod 4$.

Both groups act freely on $\mathfrak{H}$ by fractional linear transformations. The functions $k(\tau)$ and $k(\tau)^2$ induce (bi-holomorphic) homeomorphisms

$$\overline{k^2} : \mathfrak{H}/\Gamma(2) \longrightarrow \mathbb{C} \setminus \{0, 1\} = \mathcal{M}, \qquad \overline{k} : \mathfrak{H}/\Gamma_2(4) \longrightarrow \mathbb{C} \setminus \{0, \pm 1\} = \mathcal{M}_1,$$

which make the diagram

$$\begin{array}{ccc} \mathfrak{H}/\Gamma_2(4) & \xrightarrow{\overline{k}} & \mathcal{M}_1 \\ \Big\downarrow g & & \Big\downarrow \iota \\ \mathfrak{H}/\Gamma(2) & \xrightarrow{\overline{k^2}} & \mathcal{M} \end{array}$$

commutative, where g is the covering induced by the inclusion $\Gamma_2(4) < \Gamma(2)$ (i.e., by the identical map on $\mathfrak{H}$) and $\iota(z) = z^2$.

Proposition 3.8. *The virtual endomorphism ϕ of $\Gamma(2) = \pi_1(\mathcal{M})$ associated with the a.g.m. correspondence is given by*

$$\phi \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{cc} a & b/2 \\ 2c & d \end{array}\right].$$

In particular, domain of ϕ is the subgroup $\Gamma_2(4)$.

Proof. Choose a base-point $z \in \mathcal{M}$ and let $\tau \in \mathfrak{H}$ be any of its preimages under the universal covering map k^2. Then the point $z_1 = k(\tau/2) \in \mathcal{M}_1$ is an f-preimage of z, since

$$f(z_1) = f(k(\tau/2)) = k(\tau)^2 = z,$$

by (7). Connect τ to $\tau/2$ by a path $\tilde{\ell}$ and let $k^2(\tilde{\ell}) = \ell$ be its image in $\mathcal{M}$. The path ℓ connects z to $\iota(z_1) = k(\tau/2)^2$. Let us compute the virtual endomorphism

$$\phi(\gamma) = \ell\iota(f^{-1}(\gamma)_{z_1})\ell^{-1}$$

associated to the a.g.m. correspondence. Here and below we denote by $f^{-1}(\gamma)_{z_1}$ the lift of γ by f starting at z_1.

Let $\gamma \in \pi_1(\mathcal{M}, z)$ be an arbitrary element of the fundamental group. The k^2-lift of γ to $\mathfrak{H}$ starting in τ is a path $\tilde{\gamma}$ connecting τ to $\frac{a\tau+b}{c\tau+d}$, where $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ is the element of $\Gamma(2)$ identified with γ under the natural isomorphism of $\pi_1(\mathcal{M}, z)$ with $\Gamma(2)$. The curve $\tilde{\gamma}/2$ will connect the point $\tau/2$ to the point $\frac{a\tau+b}{2c\tau+2d}$. Let $\gamma_1 = k(\tilde{\gamma}/2)$. The path γ_1 starts in $k(\tau/2) = z_1$ and we have

$$f(\gamma_1) = f(k(\tilde{\gamma}/2)) = k(\tilde{\gamma})^2 = \gamma,$$

hence $\gamma_1 = f^{-1}(\gamma)_{z_1}$. We have $\iota(\gamma_1) = k(\tilde{\gamma}/2)^2$, i.e., $\tilde{\gamma}/2$ is the lift of $\iota(\gamma_1)$ by the universal covering map $k^2 : \mathfrak{H} \longrightarrow \mathcal{M}$. The end $\frac{a\tau+b}{2c\tau+2d}$ of the path $\tilde{\gamma}/2$ is obtained from its beginning $\tau/2$ by application of the linear fractional transformation $z \mapsto \frac{az+b/2}{2cz+d}$. It follows that if γ_1 is a loop, then the curve

$$\delta = \tilde{\ell} \cdot \tilde{\gamma}/2 \cdot \left(\frac{a\tilde{\ell} + b/2}{2c\tilde{\ell} + d}\right)^{-1}$$

is the k^2-lift of the loop $\ell \cdot \iota(\gamma_1) \cdot \ell^{-1} = \phi(\gamma)$, hence $\phi(\gamma)$ is identified with $\begin{bmatrix} a & b/2 \\ 2c & d \end{bmatrix}$, since $(a\tau + b/2)/(2c\tau + d)$ is the end of δ. □

Theorem 3.9. *The iterated monodromy group of the arithmetic-geometric mean is generated by*

$$\alpha = \sigma(1, \alpha), \qquad \beta = (\beta^2, (\beta^{-1}\alpha)^2),$$

and is free.

Proof. The fundamental group of $\mathcal{M}$ is freely generated by the matrices

$$\alpha = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

The domain of the virtual endomorphism ϕ of $\Gamma(2)$ is generated by the matrices

$$\alpha^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \gamma = \alpha^{-1}\beta\alpha = \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}.$$

The virtual endomorphism acts on the generators of its domain by

$$\phi(\alpha^2) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \alpha, \quad \phi(\beta) = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} = \beta^2,$$

$$\phi(\gamma) = \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix} = (\beta^{-1}\alpha)^2.$$

It follows that one of the standard actions of the iterated monodromy group is given by the wreath recursion

$$\alpha = \sigma(1, \alpha), \qquad \beta = (\beta^2, (\beta^{-1}\alpha)^2),$$

since we have then

$$\alpha^2 = (\alpha, \alpha), \qquad \alpha^{-1}\beta\alpha = ((\beta^{-1}\alpha)^2, \alpha^{-1}\beta^2\alpha),$$

hence the virtual endomorphism ϕ coincides with the projection onto the first coordinate.

Suppose that the iterated monodromy group is not free. Then there exists a normal subgroup $N \lhd \Gamma(2)$ such that N belongs to the domain $\Gamma_2(4)$ of ϕ and $\phi(N) \le N$. It follows from the formula for the virtual endomorphism ϕ that N must consist only of the fractional linear transformations of the form $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$, i.e., of fractional linear transformations fixing 0. Since N is normal, all elements of N must fix all points of the $\Gamma(2)$-orbit of 0, in particular the points 2 and -2, which are images of 0 under $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. But only the identical fractional linear transformation fixes the points 0, 2 and -2. Consequently, N is trivial. □

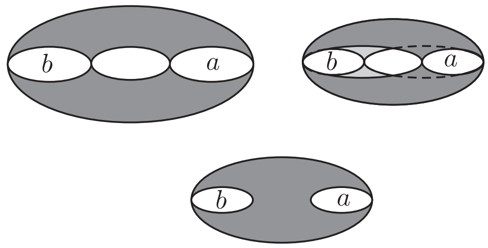

FIGURE 8. Brunner-Sidki-Vieira group

In some sense Theorem 3.9 is the most straightforward example of a self-similar free group. Other examples of self-similar free groups can be found in [GM05, VV07] and [Nek05, Subsections 1.10.2–4]. They are in some sense better, since, unlike the example from Theorem 3.9, they are generated by finite automata (i.e., for every element g of the group the set $\{g|_v \ : \ v \in \mathsf{X}^*\}$ is finite), although the proofs of faithfulness of the action are more complicated.

More examples of *critically finite* correspondences (i.e., such correspondences $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ that $\mathcal{M}$, $\mathcal{M}_1$ are punctured spheres and f and ι are coverings defined by rational functions) are given in [Bul92].

3.7.3. *Lattices in Lie groups.* The following theorem of M. Kapovich [Kap08] is closely related to the last example.

Theorem 3.10. *Let Γ be an irreducible lattice in a semisimple algebraic Lie group G. Then the following are equivalent:*

(1) Γ *is virtually isomorphic to an arithmetic lattice in G, i.e., contains a finite index subgroup isomorphic to such arithmetic lattice.*
(2) Γ *admits a faithful self-similar action which is transitive on the first level.*

The proof of the theorem is an application of the description of arithmetic lattices in terms of their commensurators by Margulis [Mar91].

All the lattices satisfying the conditions of the theorem are iterated monodromy groups of automata of the form $(G/\Gamma, G/(\phi^{-1}(\Gamma)\cap\Gamma), f, \iota)$, where $\phi : G \longrightarrow G$ is an automorphism of the Lie group G, f is the natural covering of G/Γ by $G/(\phi^{-1}(\Gamma)\cap\Gamma)$ and $\iota : G/(\phi^{-1}(\Gamma)\cap\Gamma) \longrightarrow G/\Gamma$ is the map induced by ϕ.

3.7.4. *Brunner-Sidki-Vieira group.* The *Brunner-Sidki-Vieira group* (see [BSV99]) is generated by two automorphisms of the binary tree given by the recursions

$$a = \sigma(1, a), \quad b = \sigma(1, b^{-1}), \tag{8}$$

where σ, as usual, is the transposition.

The group is torsion free and all its proper quotients are solvable. It has a finite L-presentation

$$\langle a, b \ : \ \varsigma^k([b^{-1}a, ab^{-1}]) = \varsigma^k([a^{-2}ba, aba^{-2}]) = 1, k \geq 0\rangle,$$

where the endomorphism ς of the free group $\langle a, b\rangle$ is given by

$$a \mapsto a^2, \qquad b \mapsto b^{-1}a^{-1}.$$

Figure 8 shows a topological automaton such that the Brunner-Sidki-Vieira group is its iterated monodromy group. The picture on the bottom shows the

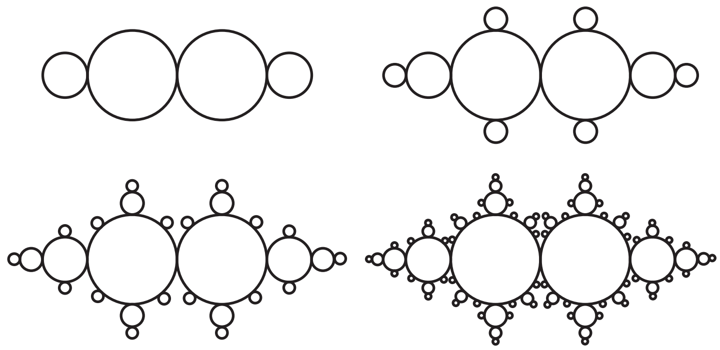

FIGURE 9. The graphs of the action of IMG $\left(-\frac{z^3}{2} + \frac{3z}{2}\right)$

space $\mathcal{M}$. It is a square with two pairs of vertices identified. The boundary loops corresponding to the generators a and b are marked by the corresponding letters.

The covering space $\mathcal{M}_1$ is shown twice on the top part of the figure. The letters mark the corresponding preimages of the loops. Both loops are lifted to a pair of non-closed paths. The right hand side picture shows how ι projects the coverings space $\mathcal{M}_1$ onto $\mathcal{M}$. The lighter shade of grey colour shows the side of the surface $\mathcal{M}_1$ opposite to the side shown on the left hand side picture. We see that ι preserves the orientation of the loop a, while it inverts the orientation of the loop b. It is easy to check that the iterated monodromy group of the defined topological polynomial is given by recursion (8).

4. Limit spaces and Julia sets

4.1. Schreier graphs.

Definition 5. Let G be a group acting on X^*. Fix a finite generating set S of G. The associated *Schreier graphs*, or *graphs of the action* of G on X^n is the graph $\Gamma_n(G, S)$ with the set of vertices X^n in which two vertices v_1, v_2 are connected by an edge if and only if there exists $s \in S$ such that $s(v_1) = v_2$.

Since the group acts on the tree X^* by automorphisms, the map $vx \mapsto v : \mathsf{X}^{n+1} \longrightarrow \mathsf{X}^n$ is a covering of the corresponding graphs.

For example, for the adding machine action we get a cycle of length 2^n, which is a double covering of the previous cycle of length 2^{n-1}.

The nth level Schreier graph of the iterated monodromy group of the Chebyshev polynomial T_d is the path of d^n vertices with loops at the ends (see Subsection 2.3). The covering of $\Gamma_n(\mathrm{IMG}\,(T_d)\,, \{a, b\})$ by $\Gamma_{n+1}(\mathrm{IMG}\,(T_d)\,, \{a, b\})$ folds the segment d times.

The Schreier graphs of the action of the generators a and b of IMG $\left(-\frac{z^3}{2} + \frac{3z}{2}\right)$ on the first four levels are shown on Figure 9.

Definition 6. Let $f(z)$ be a complex rational function. Its *Julia set* is the closure of the union of repelling cycles of f. Here a *repelling cycle* is a sequence $z_0 = f(z_n), z_1 = f(z_0), \ldots, z_n = f(z_{n-1})$ such that $|f'(z_0)f'(z_1)\cdots f'(z_n)| > 1$.

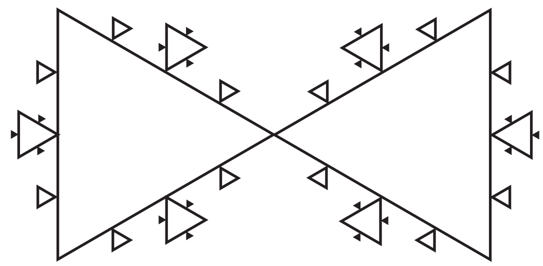

FIGURE 10. A model of the Julia set of $-\frac{z^3}{2}+\frac{3z}{2}$

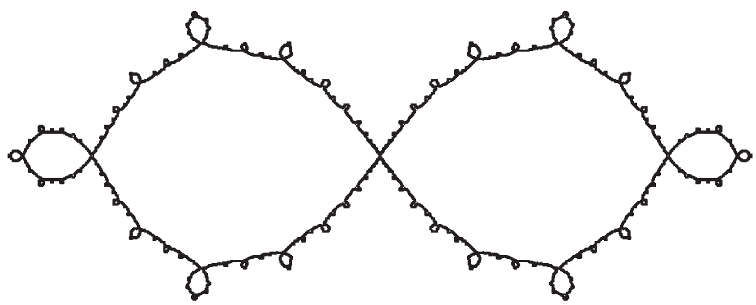

FIGURE 11. The Julia set of $-\frac{z^3}{2}+\frac{3z}{2}$

If f is a polynomial, then the Julia set of f can be equivalently defined as the boundary of the set of points z such that the sequence $(f^{\circ n}(z))_{n\ge 0}$ is bounded. For more details, see [Mil99].

Gaston Julia described a model of the Julia set of the polynomial $-\frac{z^3}{2}+\frac{3z}{2}$ in his paper [Jul18]. The model is constructed by attaching at each step regular triangles to the middles of the edges of the graph constructed on the previous step. See the fourth step of the construction on Figure 10 and compare it with the Schreier graphs of IMG $\left(-\frac{z^3}{2}+\frac{3z}{2}\right)$.

The Julia set itself is shown on Figure 11. As we will see later, the model described by G. Julia converges to the Julia set, though he is careful not to claim anything concrete about the relation of the model with the polynomial (except for some general statements about the relative arrangement of the basins of attraction), saying that it is just a scheme aiding intuition. It also seems that the only reason to use triangles was an inspiration by a recent paper of H. Koch on what is known now as the Koch curve.

4.2. **Limit space.** Previous examples suggest that the Schreier graphs of the iterated monodromy groups and the covering $\Gamma_{n+1}(G,S)\longrightarrow\Gamma_n(G,S)$ converge to some limit. This observation is formalized in the following definition.

Definition 7. Let G be a contracting self-similar group acting on X^*. Let $\mathsf{X}^{-\omega}$ be the space of left-infinite sequences $\ldots x_2x_1$, $x_i\in\mathsf{X}$, with the direct product topology. Two sequences $\ldots x_2x_1, \ldots y_2y_1\in\mathsf{X}^{-\omega}$ are equivalent if there exists a

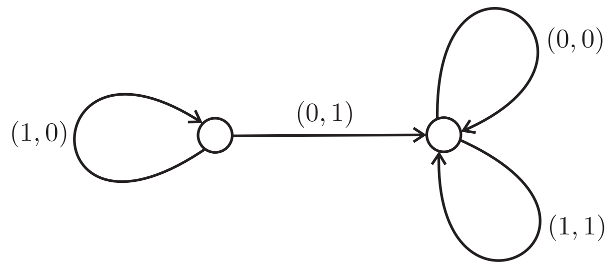

Figure 12. The Moore diagram of the adding machine

sequence g_k taking values in a finite subset of G such that

$$g_k(x_k \dots x_1) = y_k \dots y_1$$

for every k. The quotient of $\mathsf{X}^{-\omega}$ by the equivalence relation is the *limit space* $\mathcal{J}_G$.

In other words, two sequences $\dots x_2x_1$ and $\dots y_2y_1$ represent the same point of the limit space if the words $x_n \dots x_1$ and $y_n \dots y_1$ are on a uniformly bounded distance from each other in the graphs of the action of G on the levels X^n of the tree.

The shift $\dots x_2x_1 \mapsto \dots x_3x_2$ agrees with the equivalence relation, hence it induces a continuous map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$. The dynamical system $(\mathcal{J}_G, \mathsf{s})$ is called the *limit dynamical system* of the group G.

A more explicit description of the equivalence relation is given in the following proposition proved in [Nek05, Proposition 3.2.7].

Proposition 4.1. *Let G be a finitely generated contracting group acting on X^*, and let S be a finite generating set such that $g|_x \in S$ for every $g \in S$ and $x \in \mathsf{X}$. Consider the set of R_S of pairs of sequences $(\dots x_2x_1, \dots y_2y_1) \in \mathsf{X}^{-\omega} \times \mathsf{X}^{-\omega}$ for which there exists a sequence $g_n \in S$ such that*

$$g_n(x_n) = y_n, \qquad g_n|_{x_n} = g_{n-1}.$$

Then the equivalence relation generated by R_S coincides with the equivalence relation given in Definition 7.

Recall, that a *Moore diagram* of the set S from Proposition 4.1 is the oriented graph with the set of vertices S in which for every $g \in S$ and $x \in \mathsf{X}$ we have an arrow starting in g, ending in $g|_x$, and labelled by $(x, g(x)$. Then Proposition 4.1 tells us that the equivalence relation is generated by the pairs of infinite sequences read on the labels of the left-infinite oriented paths in the Moore diagram. Note that the right-infinite paths describe the action of the generators on infinite sequences. If $(x_1, y_1), (x_2, y_2), \dots$ are the labels of the edges along an oriented path starting in $g \in S$, then $g(x_1x_2\dots) = y_1y_2\dots$.

For instance, the Moore diagram of the generating set $\{1, a\}$ of the adding machine action is shown on Figure 12. We see that the left-infinite paths in the Moore diagram of $\{1, a\}$ are labelled by pairs of equal letters, or by $\dots(1, 0)(1, 0)(1, 0)$, or by

$$\dots(1, 0)(1, 0)(1, 0)(0, 1)(x_1, x_1)(x_2, x_2)\dots(x_n, x_n)$$

for some word $x_1x_2\dots x_n \in \mathsf{X}^*$ (which can be empty). Hence we get the following identifications in $\mathsf{X}^{-\omega}$:

$$\dots 1110x_1x_2\dots x_n \sim \dots 0001x_1x_2\dots x_n$$

and

$$\dots 1111 \sim \dots 0000.$$

These are the usual identifications of the real binary fractions modulo $\mathbb{Z}$. Consequently, the limit space of the adding machine action is the circle $\mathbb{R}/\mathbb{Z}$. The shift $\dots x_2x_1 \mapsto \dots x_3x_2$ coincides with the map $x \mapsto 2x$ on $\mathbb{R}/\mathbb{Z}$, hence it is a double self-covering of the circle.

4.3. **Models of the limit space.** Definition 7 is too abstract and can be used to visualize the limit space of a contracting group only in the simplest cases. The Schreier graphs are better in this sense, but it is also hard to use them to study topological properties of the limit space.

A better approach is to use approximations of the limit space by simplicial complexes. By a theorem of P. Alexandroff [Ale29], every compact metrizable finite-dimensional space $\mathcal{J}$ is an inverse limit of a sequence of finite simplicial complexes $\mathcal{M}_n$ of bounded dimension (one can take the dimension of the complexes to be equal to the dimension of the space). We will have then maps $\mathcal{J} \longrightarrow \mathcal{M}_n$ from the inverse limit to the complexes $\mathcal{M}_n$, which are becoming closer to a homeomorphism as n grows.

We want to describe a procedure producing in a simple recursive way a sequence of simplicial complexes $\mathcal{M}_n$ converging in the described sense to the limit space of a contracting group.

Definition 8. Let G be a contracting group. A *model of its limit dynamical system* is a topological automaton $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, p, \iota)$ such that $\mathcal{M}$ and $\mathcal{M}_1$ are compact (orbi)spaces with a length structure (e.g., Euclidean simplicial complexes), there exists $0 < \lambda < 1$ such that for every rectifiable curve γ in $\mathcal{M}_1$ the length of $\iota(\gamma)$ is at most λ times the length of γ, the length of $p(\gamma)$ is equal to the length of γ, and the iterated monodromy group of $\mathcal{F}$ is G.

Every model of the limit space of a contracting group provides an approximation of the limit space as an inverse limit. The following theorem is proved in [Nek08a].

Theorem 4.2. *Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, p, \iota)$ be a model of the limit space of a contracting group G. Let the spaces $\mathcal{M}_n$ and the maps $\iota_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ be defined by the pull-back diagram* (5). *Then the limit space $\mathcal{J}_G$ is homeomorphic to the inverse limit of the sequence*

$$\mathcal{M} \xleftarrow{\iota} \mathcal{M}_1 \xleftarrow{\iota_1} \mathcal{M}_2 \xleftarrow{\iota_2} \cdots .$$

We will not loose any generality in applications, if we restrict ourselves to *developable* orbispaces, i.e., to quotients of topological spaces by proper group actions. Moreover, we can reformulate Definition 8 in the following way.

Definition 9. Let $\phi : G \dashrightarrow G$ be a surjective contracting virtual endomorphism. A *topological model* of ϕ is a metric space $\mathcal{X}$ with a right proper co-compact G-action by isometries and a contracting map $F : \mathcal{X} \longrightarrow \mathcal{X}$ such that

$$F(\xi \cdot g) = F(\xi) \cdot \phi(g)$$

for all $\xi \in \mathcal{X}$ and $g \in \operatorname{Dom}\phi$.

The corresponding topological automaton is $(\mathcal{X}/G, \mathcal{X}/\operatorname{Dom}\phi, p, \iota)$, where $p : \mathcal{X}/\operatorname{Dom}\phi \longrightarrow \mathcal{X}/G$ is induced by the identity map on $\mathcal{X}$ and ι is induced by F.

Here F is called contracting if there exist $C > 0$ and $0 < \lambda < 1$ such that

$$d(F^{\circ n}(\xi_1), F^{\circ n}(\xi_2)) \leq C\lambda^n d(\xi_1, \xi_2)$$

for all $\xi_1, \xi_2 \in \mathcal{X}$, where $d(\cdot, \cdot)$ is the metric on $\mathcal{X}$.

An action of G on $\mathcal{X}$ is *proper* if for every compact set $K \subset \mathcal{X}$ the set of the elements $g \in G$ such that $K \cdot g \cap K \neq \varnothing$ is finite. It is *co-compact* if there exists a compact set $K \subset \mathcal{X}$ such that $\mathcal{X} = \bigcup_{g \in G} K \cdot g$.

Note that F induces a well-defined map $F_0 : \mathcal{X}/\ker\phi \longrightarrow \mathcal{X}$, since $F(\xi \cdot g) = F(\xi)$ for $g \in \ker\phi$. More generally, it induces maps $F_n : \mathcal{X}/\ker\phi^{n+1} \longrightarrow \mathcal{X}/\ker\phi^n$.

The group G acts on the spaces $\mathcal{X}/\ker\phi^n$ by the rule

$$(\xi \ker\phi^n) \cdot g = (\xi \cdot h)\ker\phi^n,$$

where $h \in \operatorname{Dom}\phi^n$ is such that $\phi^n(h) = g$. This action is well defined and proper.

One can show that the orbispaces $\mathcal{M}_n$ of the defined action of G on $\mathcal{X}/\ker\phi^n$ coincide with the orbispaces obtained by iteration of the associated topological automaton $(\mathcal{X}/G, \mathcal{X}/\operatorname{Dom}\phi, f, \iota)$. The maps $\iota_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ coincide with the maps induced by F_n, the coverings $p_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ will be induced by the inclusions $\ker\phi^{n+1} > \ker\phi^n$.

Theorem 4.3. *Let $F : \mathcal{X} \longrightarrow \mathcal{X}$ be a topological model of a contracting virtual endomorphism $\phi : G \dashrightarrow G$. Then the inverse limit $\mathcal{X}_G$ of the G-spaces*

$$\dots \xrightarrow{F_3} \mathcal{X}/\ker\phi^3 \xrightarrow{F_2} \mathcal{X}/\ker\phi^2 \xrightarrow{F_1} \mathcal{X}/\ker\phi \xrightarrow{F_0} \mathcal{X}$$

depends only on G and ϕ. The space of orbits $\mathcal{X}_G/G$ is the inverse limit of the spaces $\mathcal{M}_n$ with respect to the maps ι_n and is homeomorphic to $\mathcal{J}_G$.

We get the following corollary of Theorem 4.3. The proof is a rather straightforward application of the Schwarz-Pick lemma for the Poincaré metric on the *Thurston orbifold* of the rational function (see [Mil99] for these notions).

Corollary 4.4. *The limit space of* $\operatorname{IMG}(f)$ *for a post-critically finite rational function f is homeomorphic to the Julia set of f. The action of f on its Julia set is conjugate to the limit dynamical system $(\mathcal{J}_{\operatorname{IMG}(f)}, \mathsf{s})$.*

The unique inverse limit $\mathcal{X}_G$ from Theorem 4.3 is called the *limit G-space* of the group G. It can be also defined as the quotient of the space $\mathsf{X}^{-\omega} \times G$ by a naturally defined equivalence relation.

Definition 10. Let G be a contracting group acting on X^*. Two sequences $\dots x_2x_1 \cdot g$ and $\dots y_2y_1 \cdot h \in \mathsf{X}^{-\omega} \times G$ are *asymptotically equivalent* if there exists a sequence g_k taking values in a finite subset of G such that

$$g(x_k \dots x_1) = y_k \dots y_1, \quad g|_{x_k \dots x_1} g = h$$

for all k.

The quotient of $\mathsf{X}^{-\omega} \times G$ by the asymptotic equivalence relation is the limit G-space $\mathcal{X}_G$. The equivalence relation is invariant with respect to the natural right action of G on $\mathsf{X}^{-\omega} \times G$, hence we get an action of G on $\mathcal{X}_G$.

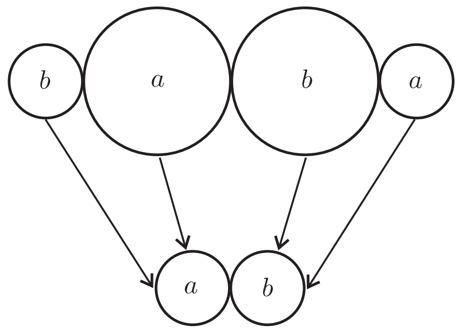

FIGURE 13. A model of the Julia set of $-\frac{z^3}{2} + \frac{3z}{2}$

For every $x \in \mathsf{X}$ we have a map on $F_x : \mathcal{X}_G \longrightarrow \mathcal{X}_G$ mapping a point represented by $\ldots x_2 x_1 \cdot g$ to the point represented by $\ldots x_2 x_1 g(x) \cdot g|_x$. This map is continuous and satisfies

$$F_x(\xi \cdot g) = F_x(\xi) \cdot \phi(g),$$

for all elements g of the stabilizer of x, where ϕ is the virtual endomorphism associated with G and x. One can also show that F_x is contracting with respect to a metric on $\mathcal{X}_G$.

4.4. **A model of the Julia set of $-\frac{z^3}{2} + \frac{3z}{2}$.** Let us apply Theorem 4.2 to the polynomial $-\frac{z^3}{2} + \frac{3z}{2}$ and show that the model of its Julia set described by G. Julia in [Jul18] is correct.

The virtual endomorphism associated with $\mathrm{IMG}\left(-\frac{z^3}{2} + \frac{3z}{2}\right)$ is

$$a^2 \mapsto a, \quad b^2 \mapsto b, \quad a^b \mapsto 1, \quad b^a \mapsto 1. \tag{9}$$

Consider the topological automaton, shown on Figure 13. Here the arrows describe the action of the map $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$. It contracts the distances on the bigger circles twice and contracts the smaller circles to single points. The covering $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is described by the labels of the circles. The bigger circles labelled by a or b cover twice the circles labelled by the corresponding letters downstairs. The smaller circles are mapped by f onto the corresponding circles homeomorphically. It is easy to check that the virtual endomorphism of the fundamental group of $\mathcal{M}$ associated with this topological automaton is (conjugate to) the endomorphism (9).

It follows that the graph $\mathcal{M}_{n+1}$ is obtained from the graph $\mathcal{M}_n$ by adding a loop in the middle of every edge.

In this case the graphs $\mathcal{M}_n$ and the canonical maps from $\lim_{\leftarrow} \mathcal{M}_n$ (i.e., from the Julia set $\mathcal{J}$ of the polynomial) to $\mathcal{M}_n$ have a nice interpretation in terms of holomorphic dynamics. *Fatou components* are the connected components of the complement of the Julia set. We can identify the graph $\mathcal{M}$ with the union of the boundaries of the Fatou components of $-z^3/2 + 3z/2$ containing the points 1 and -1 (see the Julia set of the polynomial on Figure 11). This is a subset of the Julia set $\mathcal{J}$ and is forward-invariant. The polynomial acts on each of the two circles of $\mathcal{M}$ as a double self-covering. Then the space $\mathcal{M}_n$ will be the inverse image of $\mathcal{M}$ under the action of the nth iteration of the polynomial. Thereofore, it will also be a union of a finite number of boundaries of Fatou components. The projection of $\mathcal{J}$ onto

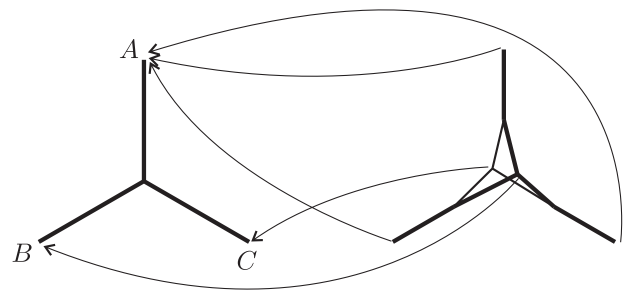

FIGURE 14. Gupta-Sidki group

$\mathcal{M}_n$ comes from the tree-like structure of the Julia set (see Figure 11). Closure of each component of $\mathcal{J} \setminus \mathcal{M}_n$ intersects with $\mathcal{M}_n$ exactly in one point, which we call the *root* of the component. The map $\mathcal{J} \longrightarrow \mathcal{M}_n$ (coming from the identification of $\lim_{\leftarrow} \mathcal{M}_n$ with the Julia set) will project the points of every component of $\mathcal{J} \setminus \mathcal{M}_n$ to its root.

4.5. **Gupta-Sidki group.** The Gupta-Sidki group (see [GS83]) is generated by two automorphisms s, t of the tree $\{0, 1, 2\}^*$, where s is the transitive cycle (012) acting only on the first letter of words and t is given by the recursion

$$t = (s, s^{-1}, t).$$

Figure 14 gives a contracting topological model $(\mathcal{M}, \mathcal{M}_1, f, \iota)$ of the limit space of the Gupta-Sidki group. In this case $\mathcal{M}$ and $\mathcal{M}_1$ are *graphs of groups*, i.e., orbispaces of action of a group on a tree. For the theory of graphs of groups, see [Ser80, BH99].

The orbispace $\mathcal{M}$, shown on the left hand side of the figure, is a tripod of groups with cyclic groups of order three at the feet A, B, C.

The covering graph of groups $\mathcal{M}_1$ is shown on the right hand side of the figure. It is obtained by gluing together three copies of the tripod ABC along two feet. The graph of groups $\mathcal{M}_1$ has three cyclic vertex groups of order 3, which are mapped by the covering to the foot A of $\mathcal{M}$. The two common points of the copies of the tripod are mapped to the feet B and C of $\mathcal{M}$. The map $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ "projects" the graph $\mathcal{M}_1$ onto the graph $\mathcal{M}$. It maps the vertices of degree one of $\mathcal{M}_1$ (i.e., the elements of the set $f^{-1}(A)$) to the feet of $\mathcal{M}$, and maps the "internal" part of $\mathcal{M}_1$ onto the inner halves of the legs of $\mathcal{M}$. The map ι will be contracting with coefficient 2.

The fundamental group $\pi_1(\mathcal{M})$ is generated by the generators a, b, c of the groups of the vertices A, B and C, respectively. The universal covering of $\mathcal{M}$ is a regular ternary tree with vertices coloured into two colours: one corresponding to the inverse images of the feet A, B, C; the other corresponding to the central point of the tripod. The fundamental group of $\mathcal{M}$ is the free product $\langle a \rangle * \langle b \rangle * \langle c \rangle$ of three cyclic groups or order 3 acting on the tree in the natural way (by "rotations" around the inverse images of A, B and C). The covering orbispace $\mathcal{M}_1$ corresponds to the subgroup of the fundamental group generated by the elements bc, cb, a, a^b, a^c, which has index three in $\pi_1(\mathcal{M})$. The loops a, a^b and a^c are lifted by the covering f to generators of the isotropy groups of the feet of $\mathcal{M}_1$. The loops a^b and a^c will be lifted to loops going around two holes of the central graph of $\mathcal{M}_1$. It follows that the virtual endomorphism associated with the described topological automaton is

$$\begin{array}{rcl} bc & \mapsto & 1, \\ a & \mapsto & a, \\ a^c & \mapsto & c. \end{array} \qquad \begin{array}{rcl} cb & \mapsto & 1, \\ a^b & \mapsto & b, \end{array}$$

Consequently, the iterated monodromy group of the automaton is the Gupta-Sidki group, where the epimorphism from the fundamental group of $\mathcal{M}$ onto the iterated monodromy group is given by

$$a \mapsto t, \qquad b \mapsto s, \qquad c \mapsto s^{-1}.$$

5. Miscellaneous examples

5.1. **Abelian groups.** Let A be an $n \times n$-matrix with integer entries and let $|\det A| = d > 1$. Then the linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ induces a d-fold self-covering of the torus $\mathbb{R}^n/\mathbb{Z}^n$.

The associated virtual endomorphism is $A^{-1} : \mathbb{Z}^n \dashrightarrow \mathbb{Z}^n$ with the domain $A(\mathbb{Z}^n)$ of index d in $\mathbb{Z}^n$.

Choosing a coset representative system (equivalently a collection of connecting paths on the torus) we get the associated iterated monodromy action on X^*, which corresponds to a *numeration system* on $\mathbb{Z}^n$.

Namely, if $r_1, \ldots, r_d$ is a coset-representative system of $\mathbb{Z}^n/A(\mathbb{Z}^n)$, then every element $x \in \mathbb{Z}^n$ is uniquely written as a formal sum

$$x = r_{i_0} + A(r_{i_1}) + A^2(r_{i_2}) + \cdots ,$$

where r_{i_k} is defined by the condition

$$x - \left(r_{i_0} + A(r_{i_1}) + \cdots + A^k(r_{i_k})\right) \in A^{k+1}(\mathbb{Z}^n).$$

The associated action of $\mathbb{Z}^n = \pi_1(\mathbb{R}^n/\mathbb{Z}^n)$ describes addition of the elements of $\mathbb{Z}^n$ to such formal series. The series are convergent in the completion of $\mathbb{Z}^n$ with respect to the sequence $(A^k(\mathbb{Z}^n))_{k\geq 0}$ of subgroups of finite index.

The following theorem is basically proved in [NS04], see also [Nek05, Proposition 2.9.2].

Theorem 5.1. *If no eigenvalue of A^{-1} is an algebraic integer, then* $\mathrm{IMG}\,(A) = \pi_1(\mathbb{R}^n/\mathbb{Z}^n) = \mathbb{Z}^n$.

Proposition 5.2. *The iterated monodromy group of $A : \mathbb{R}^n/\mathbb{Z}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n$ is contracting if and only if A is expanding (i.e., all eigenvalues are greater than* 1 *in absolute value). In this case the limit G-space $\mathcal{X}_{\mathbb{Z}^n}$ of the iterated monodromy group is the space $\mathbb{R}^n$ with the natural action of $\mathbb{Z}^n$ on it, and the limit space $\mathcal{J}_{\mathbb{Z}^n}$ is the torus $\mathbb{R}^n/\mathbb{Z}^n$.*

Proof. The first part is a direct corollary of Theorem 3.2.

For the second part we can apply Theorem 4.3. The group $\mathbb{Z}^n$ acts naturally on the space $\mathbb{R}^n$, and the action is proper and co-compact. The virtual endomorphism $A^{-1} : A(\mathbb{Z}^n) \longrightarrow \mathbb{Z}^n$ is surjective and is induced by the contracting map $A^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Therefore, the map $A^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a topological model of the virtual endomorphism (see Definition 9). Since the virtual endomorphism is injective, the inverse sequence in Theorem 4.3 is just the sequence of the spaces $\mathbb{R}^n$ and the maps A^{-1}. Consequently, the limit G-space of $\mathbb{Z}^n$ is the space $\mathbb{R}^n$. □

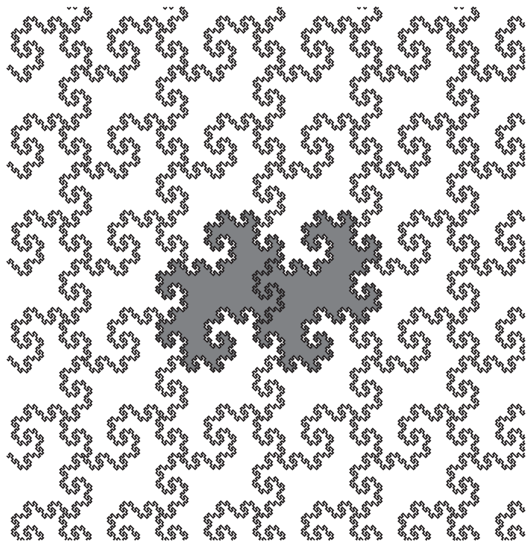

FIGURE 15. Twin dragon tiling

The points of the limit $\mathbb{Z}^n$-space $\mathbb{R}^n$ are encoded by the sequences from $\mathsf{X}^{-\omega} \cdot \mathbb{Z}^n$ according to "base A" numeration system. Namely, a sequence $\ldots i_2 i_1 \cdot g$ encodes the point

$$\xi = g + A^{-1}(r_{i_1}) + A^{-2}(r_{i_2}) + A^{-3}(r_{i_3}) + \cdots \in \mathbb{R}^n.$$

Note that convergence of the above series follows from the fact that A is expanding.

As an example, take A to be multiplication by $(i-1)$ on $\mathbb{C}$, equivalently it is the matrix $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ acting on $\mathbb{R}^2$. It will induce a degree 2 self-covering of the torus $\mathbb{C}/\mathbb{Z}[i] \approx \mathbb{R}^2/\mathbb{Z}^2$.

Choose the coset representatives 0 and 1 of $\mathbb{Z}[i]$ by the subgroup $(i-1)\mathbb{Z}[i]$. We get a "binary numeration system" on the Gaussian integers: every integer is uniquely written as a formal sum

$$z = x_0 + x_1(i-1) + x_2(i-1)^2 + x_3(i-1)^3 + \cdots,$$

(which will be actually finite in the sense that all but a finite number of coefficients x_k will be zero; but this will note be true in general (for instance, for the base $1+i$). This numeration system was probably introduced for the first time in [Pen65].

The torus $\mathbb{C}/\mathbb{Z}[i]$ is the limit space of the associated iterated monodromy group. The images of the *cylindrical sets* $\mathsf{X}^{-\omega}v$, $v \in \mathsf{X}^n$ tile the torus by "twin dragons". Similarly, the limit G-space (see Theorem 4.3) $\mathcal{X}_{\mathbb{Z}[i]}$ is homeomorphic to $\mathbb{C}$ with the natural action of $\mathbb{Z}[i]$ on it. The images of the sets $\mathsf{X}^{-\omega} \times \{g\}$ for $g \in \mathbb{Z}[i]$ tile the plane $\mathbb{C}$. A piece of this tiling is shown on Figure 15.

Note that the tiling is self-similar, i.e., if we multiply it by $(i-1)$, then every tile of the image will be a union of precisely two original tiles. See a discussion of this tiling in [Knu69] (see pp. 189–190). Such self-similar tilings and related numeration systems are objects of extensive studies, see the papers [Vin00, Ken92, Vin95] and references therein.

5.2. **Expanding endomorphisms of manifolds.** An endomorphism $f : \mathcal{M} \longrightarrow \mathcal{M}$ of a compact Riemannian manifold $\mathcal{M}$ is called *expanding* if there exist constants $c > 0$ and $\lambda > 1$ such that

$$\|Dp^{\circ n}(\vec{v})\| \geq c\lambda^n \|\vec{v}\|$$

for every tangent vector $\vec{v}$ and every $n \geq 1$. Here D is the differential. For instance, the self-covering of the torus $\mathbb{R}^n/\mathbb{Z}^n$ defined by an expanding matrix A is an expanding endomorphism of the torus.

Every expanding endomorphism is a covering, which is of finite degree by compactness. Consequently, we can define the iterated monodromy of f. It will be a contracting self-similar group, since the associated virtual endomorphism f_*^{-1} will be obviously contracting (by Theorem 3.2). Note also that the associated virtual endomorphism is injective (its inverse is the homomorphism $f_* : \pi_1(\mathcal{M}) \longrightarrow \pi_1(\mathcal{M})$ induced by f), hence the group $\pi_1(\mathcal{M})$ is of polynomial growth (by the same argument as in the proof of Theorem 3.6, see also [Fra70, Gro81a]).

Passing to the universal covering $\widetilde{\mathcal{M}}$ and lifting f to a map $F : \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$ we get a topological model $F^{-1} : \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$ of the virtual endomorphism f_*^{-1}. Since F^{-1} is a homeomorphism, we get from Theorem 4.3 that the limit G-space of the iterated monodromy group of f is $\widetilde{\mathcal{M}}$ and that the limit dynamical system of $\mathrm{IMG}\,(f)$ is $f : \mathcal{M} \longrightarrow \mathcal{M}$. We have thus proved the following result of M. Schub [Shu69].

Theorem 5.3. *An expanding endomorphism $f : \mathcal{M} \longrightarrow \mathcal{M}$ is determined uniquely, up to a topological conjugacy, by the action of the homomorphism f_* on the fundamental group $\pi_1(\mathcal{M})$.*

In fact, our Theorem 4.2 can be seen as a generalization of Shub's theorem for general topological (orbi)spaces.

M. Gromov in [Gro81a] proved that groups of polynomial growth are virtually nilpotent, thus proving a conjecture of M. Shub [Shu70, Hir70]. In our terms, the conjecture states that the limit G-space $\widetilde{\mathcal{M}}$ of $\mathrm{IMG}\,(f)$ for an expanding endomorphism $f : \mathcal{M} \longrightarrow \mathcal{M}$ is a nilpotent Lie group on which $\mathrm{IMG}\,(f) = \pi_1(\mathcal{M})$ acts properly by affine transformations. The map $F : \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$ is an expanding automorphism of the Lie group $\widetilde{\mathcal{M}}$. See a proof of this result (valid also for orbifolds) in [Nek05, Section 6.1.2]. The proof uses Gromov's theorem on groups of polynomial growth

An example (see [PN08]) of an expanding endomorphism of a nil-manifold is defined by the following expanding automorphism of the real Heisenberg group $\mathcal{X}$

$$F : \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2z & -2y + 2xz \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}.$$

Let G be the subgroup of the matrices with $x, y, z \in \mathbb{Z}$. Then G acts on $\mathcal{X}$ by multiplication from the right. Let $\mathcal{M}$ be the quotient manifold (the action of G on $\mathcal{X}$ is free, proper and co-compact). The map F induces a covering $f : \mathcal{M} \longrightarrow \mathcal{M}$ of degree 4. The group $\mathrm{IMG}\,(f)$ coincides with the fundamental group and is generated by the matrices

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The associated wreath recursion on IMG (f) is

$$a = (12)(34)(1, b, 1, b), \quad b = (24)(a, a^b, a, a).$$

A similar expanding automorphism of the Heisenberg group and the associated tiling was studied in [Gel94]. The associated wreath recursion is described in [PN08].

5.3. **Grigorchuk group and Ulam-von Neumann map.** The *Grigorchuk group* is generated by the wreath recursion

$$a = \sigma, \quad b = (a, c), \quad c = (a, d), \quad d = (1, b),$$

where σ is the transposition (12).

It is easy to see that the generators a, b, c, d are of order 2 and that $\{1, b, c, d\}$ is isomorphic to $C_2 \times C_2$.

The Grigorchuk group is a particularly easy example of an infinite finitely generated periodic group. It was defined as a group of measure-preserving transformations of the interval in [Gri80] (it appeared for the first time implicitly in [Ale72]). It is the first example of a group of intermediate growth [Gri83], and it has many other interesting properties, see [Har00, BGŠ03, Gri05].

The Grigorchuk group is contracting (which was basically proved in the original paper [Gri80]). Let us describe its limit dynamical system.

Recall that the iterated monodromy group of the Chebyshev polynomial T_2 is isomorphic to the infinite dihedral group and is generated by

$$\alpha = \sigma, \quad \beta = (\alpha, \beta),$$

see Subsection 2.5.2 (we have renamed the generators of IMG (T_2) in order not to confuse them with the generators of the Grigorchuk group, and have changed the order of the letters of the alphabet).

It is not hard to check that for every $v \in \{0, 1\}^*$ and $g \in \{b, c, d\}$ we have $a(v) = \alpha(v)$ and

$$g(v) = v, \quad \text{or} \quad g(v) = \beta(v).$$

Moreover, for every word v there exists $g \in \{b, c, d\}$ such that $g(v) = \beta(v)$.

It follows that the Schreier graphs $\Gamma_n(G)$ of the Grigorchuk group are obtained from the Schreier graphs of IMG (T_2) just by adding some loops and duplicating some of the edges. An explicit description of the Schreier graphs of the Grigorchuk group is given in [BG00].

Consequently, the asymptotic equivalence relations on $\{0, 1\}^{-\omega}$ defined by the actions of G and of IMG (T_2) coincide. Therefore the limit dynamical systems of G and IMG (T_2) also coincide (i.e., are topologically conjugate).

The limit dynamical system of IMG (T_2) is the action of the polynomial $T_2(z) = 2z^2 - 1$ on its Julia set $[-1, 1]$ (by Corollary 4.4). This dynamical system was studied by M. Ulam and J. von Neumann (in a conjugate form of $f(x) = 4x(1-x)$ acting on $[0, 1]$, see the abstract [UvN47]) and is called sometimes the "Ulam-von Neumann map".

The *tent map* $T : [0, 1] \longrightarrow [0, 1]$ is given by

$$T(x) = \begin{cases} 2x, & \text{if } x \in [0, 1/2] \\ 2 - 2x, & \text{if } x \in [1/2, 1]. \end{cases}$$

See the graphs of the Ulam-von Neumann map (left) and the tent map (right) on Figure 16.

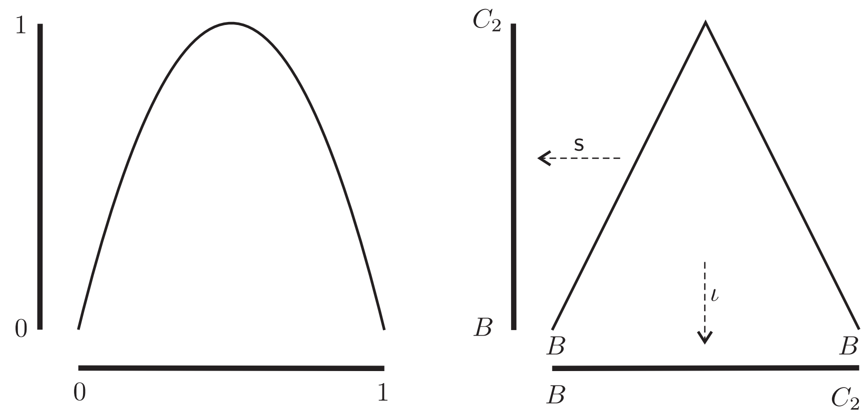

FIGURE 16. Ulam-von Neumann map and Tent map

Proposition 5.4. *The limit dynamical systems of* IMG (T_2) *and of the Grigorchuk group are conjugate to the tent map.*

As a corollary we get the classical fact that action of the Ulam-von Neumann map on $[0, 1]$ is conjugate with the tent map.

Proof. Let us use Theorem 4.3. Consider the action of the infinite dihedral group $D_\infty \cong \mathrm{IMG}\,(T_2)$ on the real line $\mathbb{R}$ by the transformations of the form $x \mapsto \pm x + n$ where n is an integer. A fundamental domain of this action is the segment $[0, 1/2]$. The stabilizers of the endpoints of this segment in D_∞ are groups of order 2. Hence, the orbispace $\mathbb{R}/D_\infty$ is a segment of two groups of order 2.

The map $x \mapsto x/2$ induces the virtual endomorphism transforming $x \mapsto \pm x + n$ to $x \mapsto \pm x + n/2$. It is easy to check that this virtual endomorphism coincides with the endomorphism

$$\beta \mapsto \beta, \quad \beta^\alpha \mapsto \alpha$$

associated with the self-similar group IMG (T_2). Here β corresponds to the transformation $x \mapsto -x$ and α to $-x+1$. Consequently, by Theorem 4.3 the limit IMG (T_2)-space is $\mathbb{R}$ with the described action. It follows that the limit space $\mathcal{J}_{\mathrm{IMG}(T_2)}$ is the orbispace $\mathbb{R}/D_\infty$ and that the limit dynamical system $\mathsf{s} : \mathcal{J}_{\mathrm{IMG}(T_2)} \longrightarrow \mathcal{J}_{\mathrm{IMG}(T_2)}$ is induced by the map $F^{-1} : x \mapsto 2x$ on $\mathbb{R}$, i.e., is the tent map. □

We see that the limit dynamical system $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ does not determine the group G, if we look at $\mathcal{J}_G$ just as at a topological space. Nevertheless, it will determine the group G, if we include the orbispace structure. Namely, for every contracting group G the limit space $\mathcal{J}_G$ is the orbispace of the action of G on the limit G-space $\mathcal{X}_G$ (see Definition 10). We get in this way (assuming that the associated virtual endomorphism ϕ is onto) a topological automaton $(\mathcal{M}, \mathcal{M}_1, \mathsf{s}, \iota)$, where $\mathcal{M} = \mathcal{X}_G/G$, $\mathcal{M}_1 = \mathcal{X}_G/\operatorname{Dom}\phi$ are orbispaces, the covering $\mathsf{s} : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is induced by the identity map on $\mathcal{X}_G$ (i.e,. by the inclusion $\operatorname{Dom}\phi < G$) and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is induced by

$$F_x(\ldots x_2 x_1 \cdot g) = \ldots x_2 x_1 g(x) \cdot g|_x$$

for any $x \in \mathsf{X}$.

The map ι is a homeomorphism of the topological spaces $\mathcal{M}$ and $\mathcal{M}_1$, but it is not an isomorphism of the orbispaces in general. It is surjective on the isotropy groups and is an *embedding* of orbispaces (see [Nek05, Section 4.6]).

The described orbispace automaton is called the *limit orbispace dynamical system* of G. One can show (see [Nek05, Theorem 5.3.1]) that G is its iterated monodromy group. Thus, the limit orbispace dynamical system $(\mathcal{J}_G, \mathsf{s})$ determines G in a unique way.

We have seen above that for $G = \mathrm{IMG}\,(T_2)$ the orbispace $\mathcal{M}$ is the segment $\mathbb{R}/D_\infty$ of two groups of order two, where D_∞ acts on $\mathbb{R}$ by the transformations $x \mapsto \pm x + n$, $n \in \mathbb{Z}$. The orbispace $\mathcal{M}_1$ is the orbispace for the action of the subgroup of the transformations with even n. Hence, $\mathcal{M}_1$ is also a segment of groups of order two. The covering $\mathsf{s} : \mathcal{M}_1 \longrightarrow \mathcal{M}$ folds the segment $\mathcal{M}_1$ in two. The map $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is induced by the map $x \mapsto x/2$ of $\mathbb{R}$, hence it is an isomorphism of the orbispaces. We can identify then $\mathcal{M}_1$ and $\mathcal{M}$ by the isomorphism ι. Then $\mathsf{s} : \mathcal{M} \longrightarrow \mathcal{M}$ becomes the tent map seen as a self-covering of the segment of groups of order two. The dihedral group is the fundamental group of the graph of groups $\mathcal{M}$ and it coincides with the iterated monodromy group of the self-covering $\mathsf{s} : \mathcal{M} \longrightarrow \mathcal{M}$.

We get a different limit orbispace dynamical system for the Grigorchuk group (but the same topological limit dynamical system). The limit orbispace $\mathcal{M} = \mathcal{J}_G$ of the Grigorchuk group is a segment of groups $C_2 \times C_2$ and C_2. We will identify the segment with $[0, 1] \subset \mathbb{R}$ so that $C_2 \times C_2$ and C_2 are the isotropy groups of the points 0 and 1, respectively. The fundamental group of this orbispace (graph of groups) is the free product $(C_2 \times C_2) * C_2$.

The covering orbispace $\mathcal{M}_1$ is the segment of two copies of $C_2 \times C_2$ (also identified with $[0, 1]$) with the covering map $\mathsf{s} : \mathcal{M}_1 \longrightarrow \mathcal{M}$ acting as the tent map T on the underlying space $[0, 1]$. The morphism $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ acts as the identical (on $[0, 1]$) homeomorphism of the segments, maps the isotropy group $C_2 \times C_2$ of 1 in $\mathcal{M}_1$ surjectively onto the isotropy group C_2 of 1 in $\mathcal{M}$ and induces an isomorphism of the isotropy groups $C_2 \times C_2$ of 0. See a schematic description of the maps s and ι on the right-hand side of Figure 16. The graph of the tent map represents $\mathcal{M}_1$, the segments on the coordinate axes represent $\mathcal{M}$, letter B represents the isotropy group $C_2 \times C_2$.

We can identify the isotropy groups $C_2 \times C_2$ and C_2 of the ends of $\mathcal{M}$ with the subgroups $\{1, b, c, d\}$ and $\{1, a\}$ of G, respectively, in such a way that ι induces the maps

$$b_1 \mapsto a, \quad c_1 \mapsto a, \quad d_1 \mapsto 1,$$

and

$$b_0 \mapsto c, \quad c_0 \mapsto d, \quad d_0 \mapsto b,$$

where g_x for $x \in \{0, 1\}$ is the lift by the covering s of the element g of the isotropy group of the end 1 of $\mathcal{M}$ to an element of the isotropy group of the point x of $\mathcal{M}_1$.

It is checked directly that the iterated monodromy group of the described orbispace automaton is the Grigorchuk group.

A natural question arises: what other contracting groups have the tent map as their limit dynamical system? In other words: what are all possible iterated monodromy groups of the tent map?

Definition 11. Let $p(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$ be a polynomial over $\mathbb{Z}/2\mathbb{Z}$. Define a self-similar group $G_{p(x)}$ generated by subgroups $B \cong C_2^m$ and $A \cong C_2 =$

$\{1, a\}$, acting on the tree $\{1, 2\}^*$ according to the wreath recursion

$$a = (12), \quad b = (\iota_1(b), \iota_2(b)),$$

for all $b \in B$, where $\iota_1 : B \longrightarrow A$ is the epimorphism given by the matrix

$$[0 \quad 0 \quad \dots \quad 0 \quad 1]$$

and $\iota_2 : B \longrightarrow B$ is the automorphism given by the matrix

$$\begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \dots & 0 & a_1 \\ 0 & 1 & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{m-1} \end{bmatrix}.$$

Theorem 5.5 (Z. Sunić and V. Nekrashevych)**.** *The limit dynamical system $(\mathcal{J}_G, \mathsf{s})$ of a contracting group G is topologically conjugate to the tent map if and only if G is equivalent as a self-similar group to the group $G_{p(x)}$ for some polynomial $p(x)$ over $\mathbb{Z}/2\mathbb{Z}$.*

The dihedral group $D_\infty = \mathrm{IMG}\,(T_2)$ is the group G_{x+1}. The Grigorchuk group is the group G_{x^2+x+1}. Another group of this family which was studied before is the group G_{x^2+1}. It is one of the groups (denoted $G_{010101\dots}$) defined by R. Grigorchuk in [Gri85]. It was later studied by A. Erschler in [Ers04], where she proved that the growth function of G_{x^2+1} is bounded below by $\exp\left(\frac{n}{\log^{2+\epsilon}(n)}\right)$ and above by $\exp\left(\frac{n}{\log^{1-\epsilon}(n)}\right)$ for all $\epsilon > 0$ and all sufficiently big n. The same group was used in [Nek10] to construct a group of non-uniform exponential growth.

The proof Theorem 5.5 is rather straightforward. It is not hard to see that the limit orbispace $\mathcal{J}_G$ has to be a segment of groups and that the isotropy group of one of the ends (the one corresponding to 1 for the standard tent map $T : [0, 1] \longrightarrow [0, 1]$) is $A = C_2$. Let B be the isotropy group of the other end. Then $\mathcal{M}_1$ is a segment connecting two copies of B. The morphism $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ will induce an epimorphism $B \longrightarrow A$ on one end and an isomorphism $B \longrightarrow B$ on the other (see the right-hand side of Figure 16). We conclude that the iterated monodromy group of the defined orbispace automaton is given by the wreath recursion described in the theorem. It remains to show that B is an elementary abelian 2-group, which easily follows from the recursion (we assume that G acts faithfully on the tree). The matrix form of the morphisms ι_i also follows from faithfulness of the action and is proved in [Šun07, Proposition 2].

The polynomial $p(x)$ is the minimal polynomial of the automorphism ι_2, and we get in this way a bijection $p(x) \mapsto G_{p(x)}$ between the set of polynomials over $\mathbb{Z}/2\mathbb{Z}$ and the set of contracting groups with the limit dynamical system conjugate with the tent map. The family of groups $G_{p(x)}$ was defined by Z. Šunić before its connection with the tent map was established. He proved in [Šun07] the following properties of the groups $G_{p(x)}$.

Theorem 5.6. *If $p_1(x)$ is divisible by $p_2(x)$, then $G_{p_1(x)} \geq G_{p_2(x)}$. If $p(x)$ is not divisible by $x + 1$, then $G_{p(x)}$ is a 2-group. If $p(x) \neq x + 1$, then $G_{p(x)}$ has intermediate growth, and its closure in the automorphism group of the tree has Hausdorff dimension $1 - \frac{3}{2^{\deg p + 1}}$.*

Here the Hausdorff dimension of a closed subgroup G of the automorphism group of the rooted binary tree is defined as

$$\liminf_{n\to\infty} \frac{1}{2^n - 1} \log_2 [G : G_n],$$

where G_n is the stabilizer of the nth level of the tree in G.

5.4. **Quadratic polynomials.** Iterated monodromy groups of post-critically finite polynomials (i.e., the corresponding wreath recursions) are described in [Nek05] and [Nek09]. Iterated monodromy groups of quadratic polynomials are studied in more detail in the paper [BN08]. We present here a survey of its results.

Every quadratic polynomial is conjugate (by an affine transformation) with a polynomial of the form $z^2 + c$. Let us describe a parametrization of the post-critically finite quadratic polynomials by rational angles $\theta \in \mathbb{R}/\mathbb{Z}$.

The *Mandelbrot set* (see [Man80, DH84]) is the set M of numbers $c \in \mathbb{C}$ such that the sequence

$$0, f(0), f^{\circ 2}(0), f^{\circ 3}(0), \ldots$$

is bounded, where $f(z) = z^2 + c$.

We remind here the main facts of the theory of *external rays* to the Mandelbrot set. Details and proofs can be found in the original manuscript [DH84, DH85].

There exists a unique bi-holomorphic isomorphism $\Phi : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \mathbb{C} \setminus M$ tangent to identity at infinity. Here $\overline{\mathbb{D}} = \{z \in \mathbb{C} \ : \ |z| \le 1\}$.

The image R_θ of the ray $\{r \cdot e^{\theta \cdot 2\pi i} \ : \ r \in (1, +\infty)\}$ under Φ is called the *external (parameter) ray at the angle* θ. We say that a ray R_θ *lands* on a point $c \in M$ if $c = \lim_{r \searrow 1} \Phi\left(r \cdot e^{\theta \cdot 2\pi i}\right)$. It is known that rays with $\theta \in \mathbb{Q}/\mathbb{Z}$ land.

If the orbit $\{f^{\circ n}(c)\}_{n \ge 1}$ of c is *pre-periodic* (i.e., if $f^{\circ n}(c) = f^{\circ m}(c)$ for some $n < m$, but $f^{\circ n}(c) \ne c$ for any n), then c belongs to the boundary of M and it is a landing point of a finite number of external rays R_θ. Each such angle θ is a rational number with even denominator.

In the other direction, if $\theta \in \mathbb{Q}/\mathbb{Z}$ has even denominator, then the ray R_θ lands on a point $c_\theta \in M$ such that the orbit of c_θ under action of $f(z) = z^2 + c_\theta$ is pre-periodic.

For example, the landing point of $R_{1/6}$ is i. The orbit of i under $z^2 + i$ is $i \mapsto -1 + i \mapsto -i \mapsto -1 + i$. The orbit of $1/6$ under angle doubling is $1/6 \mapsto 1/3 \mapsto 2/3 \mapsto 4/3 = 1/3$.

If c is periodic (i.e, if $f^{\circ n}(c) = c$ for some n), then c is an internal point of M. There are two rays $R_{\theta_1}, R_{\theta_2}$ landing on the *root* of the component of the interior of M to which c belongs. Both angles θ_i have odd denominators and their periods under the angle doubling map are equal to the period of c under the action of $z^2 + c$.

If $\theta \in \mathbb{Q}/\mathbb{Z}$ has odd denominator, then R_θ lands on a root of a component of the interior of M such that the centre c_θ of this component is a point periodic under the action of $z^2 + c_\theta$. The period of c_θ will coincide with the period of θ under the angle doubling map. We have not defined precisely the notions of the root and centre of a (hyperbolic) component of the interior of M, but the only important fact for us is that every angle $\theta \in \mathbb{Q}/\mathbb{Z}$ with odd denominator determines a post-critically finite polynomial $z^2 + c_\theta$.

For example, the orbit of -1 under $z^2 - 1$ is $-1 \mapsto 0 \mapsto -1$. The corresponding angles are $1/3$ and $2/3$. The action of angle doubling is $1/3 \mapsto 2/3 \mapsto 4/3 = 1/3$.

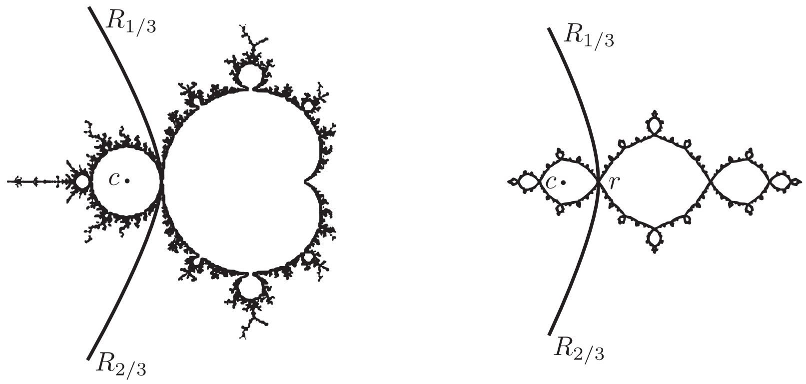

FIGURE 17. External rays

See Figure 17 for the external rays $R_{1/3}$ and $R_{2/3}$ to the Mandelbrot set (on the left-hand side) and to the Julia set of z^2-1 (on the right-hand side).

Fix $\theta \in \mathbb{Q}/\mathbb{Z}$. The points $\theta/2$ and $(\theta+1)/2$ divide the circle $\mathbb{R}/\mathbb{Z}$ into two open semicircles S_0, S_1. Here S_0 is the semicircle containing 0.

The *kneading sequence* $\widehat{\theta}$ of θ is the sequence $x_1x_2\ldots$, where

$$x_k = \begin{cases} 0 & \text{if } 2^k\theta \in S_0 \\ 1 & \text{if } 2^k\theta \in S_1 \\ * & \text{if } 2^k\theta \in \{\theta/2, (\theta+1)/2\} \end{cases}$$

The iterated monodromy group of z^2+c_θ will be defined in terms of the kneading sequence $\widehat{\theta}$.

Denote for $v = x_1 \ldots x_{n-1} \in \{0,1\}^*$ by $\mathfrak{K}(v)$ the group generated by

$$a_1 = \sigma(1, a_n), \qquad a_{i+1} = \begin{cases} (a_i, 1) & \text{if } x_i = 0, \\ (1, a_i) & \text{if } x_i = 1, \end{cases}$$

Denote for non-empty $w = y_1 \ldots y_k \in \{0,1\}^*$ and $v = x_1 \ldots x_n \in \{0,1\}^*$ such that $y_k \neq x_n$ by $\mathfrak{K}(w,v)$ the group generated by

$$b_1 = \sigma, \qquad b_{j+1} = \begin{cases} (b_j, 1) & \text{if } y_j = 0 \\ (1, b_j) & \text{if } y_j = 1 \end{cases}$$

$$a_1 = \begin{cases} (b_k, a_n) & \text{if } y_k = 0 \text{ and } x_n = 1, \\ (a_n, b_k) & \text{if } y_k = 1 \text{ and } x_n = 0, \end{cases} \qquad a_{i+1} = \begin{cases} (a_i, 1) & \text{if } x_i = 0 \\ (1, a_i) & \text{if } x_i = 1 \end{cases}$$

The following description of the iterated monodromy groups of quadratic polynomials is given in [BN08].

Theorem 5.7. *Denote by z^2+c_θ the polynomial corresponding to the angle $\theta \in \mathbb{Q}/\mathbb{Z}$.*

If $\widehat{\theta} = (x_1x_2\ldots x_{n-1}*)^\infty$, then

$$\mathrm{IMG}\left(z^2+c_\theta\right) = \mathfrak{K}\left(x_1x_2\ldots x_{n-1}\right).$$

If $\widehat{\theta} = y_1y_2\ldots y_k(x_1x_2\ldots x_n)^\infty$, then

$$\mathrm{IMG}\left(z^2+c_\theta\right) = \mathfrak{K}\left(y_1y_2\ldots y_k, x_1x_2\ldots x_n\right).$$

For example, if we take $\theta = 1/3$, then $\widehat{1/3} = (1*)^\infty$ and hence $\mathrm{IMG}\left(z^2 - 1\right)$ is generated by

$$a_1 = \sigma(1, a_2), \qquad a_2 = (1, a_1).$$

Not all groups $\mathfrak{K}(v)$ and $\mathfrak{K}(w, v)$ are iterated monodromy groups of quadratic polynomials (since not all sequences are kneading sequences of some angles). For all v, for all non-empty w and for all non-periodic non-empty u with the first letter different from the first letter of w, the groups $\mathfrak{K}(v)$ and $\mathfrak{K}(w, u)$ are the iterated monodromy groups of polynomials of degree 2^k for some positive integer k (see [Nek05, Theorem 6.10.8]).

Note that the groups $\mathfrak{K}(0, 1^n)$ coincide with the groups G_{x^n+1} defined in 5.3. They are not iterated monodromy groups of any complex polynomials for $n > 1$.

Two examples of iterated monodromy groups, corresponding to smooth Julia sets, are classical groups: for $\theta = 0$ we have $\mathrm{IMG}\left(z^2\right) = \mathfrak{K}(\varnothing) = \mathbb{Z}$, and for $\theta = 1/2$ the iterated monodromy group $\mathrm{IMG}\left(z^2 - 2\right) = \mathfrak{K}(1, 0)$ is the infinite dihedral group (see Subsection 2.3).

All the remaining examples are not finitely presented. However, there are nice recursive *L-presentations*. Fix $v = x_1 \dots x_{n-1}$. Define the following endomorphism of the free group:

$$\varphi(a_n) = a_1^2, \qquad \varphi(a_i) = \begin{cases} a_{i+1} & \text{if } x_i = 0 \\ a_{i+1}^{a_1} & \text{if } x_i = 1 \end{cases}$$

Let R be the set of commutators

$$\left[a_i, a_j^{a_1^k}\right],$$

where $2 \le i, j \le n$, and $k = 0, 2$ if $x_{i-1} \ne x_{j-1}$ and $k = 1$ if $x_{i-1} = x_{j-1}$.

The following result is proved in [BN08].

Theorem 5.8. *The group $\mathfrak{K}(v)$ is has presentation*

$$\mathfrak{K}(v) = \left\langle a_1, \dots, a_n \,\middle|\, \varphi^\ell(R) \text{ for all } \ell \ge 0 \right\rangle.$$

The groups $\mathfrak{K}(w, v)$ also have finite L-presentations, but they are a bit more complicated.

Corollary 5.9. *Let $v = x_1 x_2 \dots x_{n-1} \in \{0, 1\}^*$. Write $p(t) = x_{n-1} t + x_{n-2} t^2 + \dots + x_1 t^{n-1} \in \mathbb{Z}[t]$. Then the group $\mathfrak{K}(v)$ is isomorphic to the subgroup generated by the elements $a, a^t, \dots, a^{t^{n-1}}$ of the finitely presented group*

$$\left\langle a, t \,\middle|\, a^{t^n - 2a^{p(t)}}, \left[a^{t^i}, a^{t^j a}\right], \left[a^{t^i}, a^{t^j a^3}\right] \text{ for all } 1 \le i, k < n \right\rangle.$$

Problem. Find similar embeddings for other iterated monodromy groups and their relation with the topology of the respective maps.

The following theorem is proved in [Nek05, Theorem 3.11.3].

Theorem 5.10. *Let f_1 and f_2 be post-critically finite quadratic polynomials. If $\mathrm{IMG}(f_1)$ and $\mathrm{IMG}(f_2)$ are isomorphic as abstract groups, then the Julia sets of f_1 and f_2 are homeomorphic.*

Consider, for example, the groups

$$G_1 = \langle a_1 = \sigma(1, c_1), \quad b_1 = (1, a_1), \quad c_1 = (1, b_1) \rangle = \mathfrak{K}(11),$$

$$G_2 = \langle a_2 = \sigma(1, c_2), \quad b_2 = (1, a_2), \quad c_2 = (b_2, 1) \rangle = \mathfrak{K}(10).$$

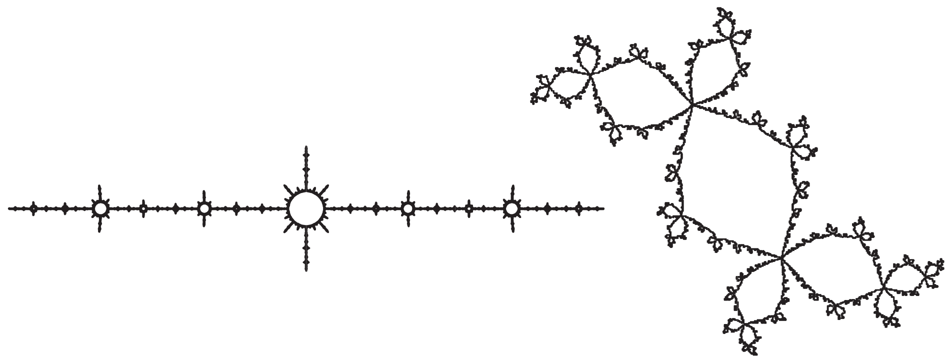

Figure 18. Aeroplane and Rabbit

They are the iterated monodromy groups of two polynomials with critical point of period 3:

$$z^2 - 0.1226\ldots + 0.7449\ldots i, \qquad z^2 - 1.7549\ldots.$$

They are not isomorphic, since the Julia sets of these polynomials (known as "Douady Rabbit" and "Aeroplane", see Figure 18) are not homeomorphic. One of the ways to show this is to prove that the Rabbit can be disconnected into three pieces by removing a point, while the Aeroplane can be disconnected only into two pieces.

It seems to be rather hard to prove that the groups G_1 and G_2 are not isomorphic by "algebraic" means. The following properties of these groups follow from the results of the paper [Nek07b].

Theorem 5.11. *The closures of the groups G_1 and G_2 in the automorphism group of the binary tree coincide.*

For every finite sets of relations and inequalities between the generators a_1, b_1, c_1 of G_1 there exists a generating set a_1', b_1', c_1' of G_2 satisfying the same relations and inequalities.

Nevertheless, there is a property that conjecturally distinguishes the groups G_1 and G_2. Namely, the elements a_1, b_1, c_1 generate a free monoid, but we do not know any example of a free subsemigroup in G_2.

Problem. Does there exists a free sub-semigroup in G_2?

The proof of the fact that a_1, b_1, c_1 generate a free monoid is straightforward, if we look at the structure of the graph of the action of G_1 on the orbit of the infinite sequence $111\ldots$, which is shown on the left-hand side part of Figure 19. The graph consists of six infinite rays decorated by finite graphs. It follows that if we apply a product of the generators a_1, b_1, c_1 to the sequence $111\ldots$, then we will know what was the first generator applied to the sequence just looking at ray to which the image of the sequence belongs. This implies that the monoid generated by a_1, b_1, c_1 is free.

The large-scale structure of the graph of the action of G_1 on the orbit of $111\ldots$ is the same as the structure of the "zoom" of the Julia set of the Douady Rabbit polynomial at the fixed point $\approx -0.2763 + 0.4797i$, which corresponds to the point

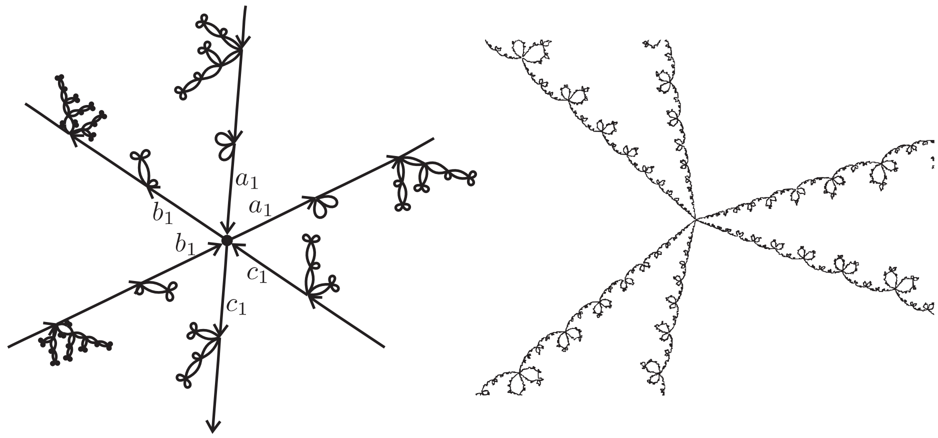

FIGURE 19. An infinite Schreier graph and a zoom of the Julia set

of the limit space of G_1 encoded by the sequence $\ldots 111$ (see the right-hand side part of Figure 19).

A more rigorous argument can be used to prove the following theorem.

Theorem 5.12. *Let f be a post-critically finite polynomial. If there exist two finite Fatou components of f with intersecting closures, then* $\mathrm{IMG}\,(f)$ *contains a free subsemigroup.*

Recall that a Fatou component is a connected component of the complement of the Julia set. One of the Fatou components is infinite (its closure is not compact): it is the basin of attraction of infinity. The remaining Fatou components are called finite.

More examples of iterated monodromy groups of exponential growth are provided by the *tuning* procedure (see [Haï00] and [Nek08c, Subsection 5.5]), which for a given polynomial f produces polynomials g such that $\mathrm{IMG}\,(f) < \mathrm{IMG}\,(g)$. In many cases the polynomial g will not satisfy the conditions of Theorem 5.12, but then $\mathrm{IMG}\,(g)$ will still contain a free subsemigroup.

Theorem 5.12 does not help us to find a free subsemigroup of the iterated monodromy group G_2 of the Aeroplane polynomial. Any two finite Fatou components of the Aeroplane polynomial have disjoint closures. Actually, we even do not know what is the growth of G_2.

There are examples of iterated monodromy groups of intermediate growth. The following is a result of K.-U. Bux and R. Perez [BP06].

Theorem 5.13. $\mathrm{IMG}\,\left(z^2+i\right)$ *has intermediate growth.*

An earlier example of an iterated monodromy group of intermediate growth is the Gupta-Fabrikowski group [FG91], which is isomorphic to $\mathrm{IMG}\,\left(z^3(-\frac{3}{2}+i\frac{\sqrt{3}}{2})+1\right)$.

Problem. Which polynomials have iterated monodromy groups of intermediate growth?

In all known cases of iterated monodromy groups $\mathrm{IMG}\,(f)$ of intermediate growth the Julia set of f is a dendrite (an $\mathbb{R}$-tree) and the post-critical points of f do not disconnect the Julia set.

5.5. **An example of Fornæss and Sibony.** We describe here some results of the papers [Nek07b, Nek10, Nek08b]. J. E. Fornæss and N. Sibony studied in [FS92] the following endomorphism of $\mathbb{PC}^2$:

$$F : [z : p : u] \mapsto [(p - 2z)^2 : (p - 2u)^2 : p^2],$$

or, in affine coordinates

$$F : (z, p) \mapsto \left(\left(1 - \frac{2z}{p}\right)^2, \quad \left(1 - \frac{2}{p}\right)^2\right).$$

The same map has appeared in a natural way in the paper [BN06] in relation with Teichmüller theory of polynomials.

Note that the second coordinate of the value of $F(z, p)$ depends only on the second coordinate p of the argument. The first coordinate of $F(z, p)$ is a quadratic polynomial in z depending on the parameter p. This *skew product* structure of F greatly facilitates its study. We will also see below how it is reflected in the structure of the iterated monodromy group of F.

The critical locus of F is the union of the lines $p = 2z$, $p = 2u$ and $p = 0$. We have

$$\{p = 2z\} \mapsto \{z = 0\} \mapsto \{z = u\} \mapsto \{z = p\} \mapsto \{z = u\},$$
$$\{p = 2u\} \mapsto \{p = 0\} \mapsto \{u = 0\} \mapsto \{p = u\} \mapsto \{p = u\},$$

hence the post-critical set is the union of the lines $z = 0, z = 1, z = p, p = 0, p = 1$ and the line at infinity.

The following theorem is proved in [Nek08b].

Theorem 5.14. *The iterated monodromy group of the map F is generated by*

$$\alpha = (\mathit{12})(\mathit{34}), \quad \beta = (\alpha, \gamma, \alpha, \gamma^\beta), \quad \gamma = (\beta, 1, 1, \beta),$$
$$T = (R, R, T, T), \quad S = (\mathit{13})(\mathit{24})(1, \beta, 1, \beta),$$

where $R = \beta\alpha\beta\gamma\beta T^{-1}S^{-1}$.

The subgroup $\langle\alpha, \beta, \gamma\rangle$ is isomorphic to the group Γ generated by

$$\alpha = (\mathit{12})(\mathit{34}), \quad \beta = (\alpha, \gamma, \alpha, \gamma), \quad \gamma = (\beta, 1, 1, \beta). \tag{10}$$

Theorem 5.15. *Denote $E_0 = \{1, 2\}$ and $E_1 = \{3, 4\}$. The subgroup $\Gamma = \langle\alpha, \beta, \gamma\rangle$ of $\mathrm{IMG}\,(F)$ is normal and coincides with the subgroup of $\mathrm{IMG}\,(F)$ leaving the subtrees*

$$T_{i_1 i_2 \ldots} = \bigcup_{n \geq 0} E_{i_1} E_{i_2} \cdots E_{i_n}$$

invariant.

The quotient $\mathrm{IMG}\,(F)\,/\Gamma$ is isomorphic to $\mathrm{IMG}\left(\left(1 - \frac{2}{p}\right)^2\right)$. It is isomorphic to the group of isometries of the lattice $\mathbb{Z}^2$.

The canonical epimorphism $\mathrm{IMG}\,(F) \longrightarrow \mathrm{IMG}\,(F)\,/\Gamma \cong \mathrm{IMG}\left((1 - 2/p)^2\right)$ is induced (due to functoriality of the iterated monodromy construction, see [Nek08c]) by the projection $(z, p) \mapsto p$ onto the second coordinate, which transforms the map $F : (z, p) \mapsto \left((1 - 2z/p)^2, (1 - 2/p)^2\right)$ into the map $p \mapsto (1 - 2/p)^2$. The kernel Γ of the epimorphism is related hence with the first coordinate of the function F. Recall that on the first coordinate of F we have quadratic polynomials depending on a parameter p. This leads to the following generalization of iterations of a post-critically finite polynomial.

Definition 12. A sequence

$$\mathbb{C} \xleftarrow{f_1} \mathbb{C} \xleftarrow{f_2} \mathbb{C} \xleftarrow{f_3} \cdots$$

of polynomials is *post-critically finite* if there is a finite set $P \subset \mathbb{C}$ such that for every n the set of critical values of $f_1 \circ f_2 \circ \cdots \circ f_n$ is contained in P.

Examples of post-critically finite sequences of polynomials are constant sequences of post-critically finite polynomials or any sequence of polynomials z^2 and $1 - z^2$. The latter is post-critically finite, since both polynomials leave the set $\{0, 1\}$ invariant, and this set contains the critical values of both polynomials.

An example of a post-critically finite sequence of polynomials is the first coordinate of iterations of the function F, namely any sequence

$$\mathbb{C} \xleftarrow{f_{p_1}} \mathbb{C} \xleftarrow{f_{p_2}} \mathbb{C} \xleftarrow{f_{p_3}} \cdots \tag{11}$$

such that $f_{p_k}(z) = \left(1 - \frac{2z}{p_k}\right)^2$ and $p_k = \left(1 - \frac{2}{p_{k+1}}\right)^2$ for all k. The critical value of $f_p(z) = (1 - 2z/p)^2$ is 0, $f_p(0) = 1$ and $f_p(1) = (1 - 2/p)^2$. Therefore, the set $P = \{0, 1, p_1\}$ satisfies the conditions of Definition 12 for the sequence f_{p_k}.

Iterated monodromy groups of post-critically finite sequences of polynomials are defined in the same way as the iterated monodromy groups of a single post-critically finite polynomial. If P is as in Definition 12, then the fundamental group $\pi_1(\mathbb{C} \setminus P, t)$ acts naturally on the tree of preimages $\bigsqcup_{n \geq 0} (f_1 \circ f_2 \circ \cdots \circ f_n)^{-1}(t)$. The iterated monodromy groups of post-critically finite sequence of polynomials are described in [Nek09].

We are ready now to interpret the iterated monodromy groups of the sequences of the form (11) in terms of the iterated monodromy group of F. Denote by $\mathcal{D}_w$ the quotient of the group Γ, given by the wreath recursion (10), by the kernel of the action on the tree T_w (see Theorem 5.15). Then the groups $\mathcal{D}_w$ are the iterated monodromy groups of the polynomial iterations (11).

Denote by Γ_w, $w \in \{0, 1\}^\infty$ the quotient of Γ by the subgroup of elements acting trivially on a neighbourhood of ∂T_w in $\partial \mathsf{X}^*$. The group Γ_w is different from $\mathcal{D}_w$ if and only if the sequence w is cofinal with $1111\ldots$. We get an uncountable family $\{\Gamma_w\}$ of three-generated groups. The set of all quotients of the free group generated by tree elements x, y, z is naturally identified with the set

$$\mathcal{G}_3 = \{(H, x, y, z) \ : \ H = \langle x, y, z \rangle\}$$

of three-generated groups H with marked generating set. There is a natural topology on $\mathcal{G}_3$ in which two groups are close if big balls around the identity in the marked Cayley graphs of the groups coincide (see [Gri85, Gro81a]). Equivalently, it is the restriction of the direct product topology on the set 2^{F_3} of subsets of the free group $F_3 = \langle x, y, z \ : \ \varnothing \rangle$ onto the subspace of normal subgroups, which is naturally identified with $\mathcal{G}_3$.

Theorem 5.16. *The map $w \mapsto (\Gamma_w, \alpha, \beta, \gamma)$ is a homeomorphism of the Cantor set $\{0, 1\}^\infty$ with a subset of the space of 3-generated groups. Two groups Γ_{w_1} and Γ_{w_2} are isomorphic if and only if w_1 and w_2 are co-final.*

For any marked group G and a finite generating set S define the *exponent of growth* as

$$e_{(G,S)} = \lim_{n\to\infty} \sqrt[n]{|B(n)|},$$

where $B(n)$ is the set of elements of G which are products of at most n generators $g \in S \cup S^{-1}$. The group G has exponential growth if and only if $e_{(G,S)} > 1$. A group is said to be of *non-uniform exponential growth* if it is of exponential growth, but infimum of $e_{G,S}$ for all finite generating sets S is equal to one.

Proposition 5.17. *The group $\Gamma_{000\ldots} = \mathcal{D}_{000\ldots}$ is* $\mathrm{IMG}\left(z^2+i\right)$ *and thus has intermediate growth. The group $\Gamma_{111\ldots}$ is an extension of C_4^∞ by a Grigorchuk group $\mathcal{D}_{111\ldots}$ and contains the lamplighter group, hence is of exponential growth.*

Here $\mathcal{D}_{111\ldots}$ coincides with the group G_{x^2+1} from the family of iterated monodromy groups of the tent map, described in Subsection 5.3, and with the group $\mathfrak{K}(0, 11)$.

It follows from Theorem 5.16 that the sequence $\Gamma_{0^n 111\ldots}$ converges to $\Gamma_{000\ldots}$ as $n \to \infty$. One can show that the function $(G, x, y, z) \mapsto e_{G,\{x,y,z\}}$ from $\mathcal{G}_3$ to $\mathbb{R}$ is upper semi-continuous. We get hence the following corollary of the last theorem.

Corollary 5.18. *The group $\Gamma_{111\ldots}$ has non-uniform exponential growth.*

The question of existence of groups of non-uniform exponential growth was asked by M. Gromov in [Gro81b, Remark 5.2] (see also [Gro99, Remark 5.B.5.12] and a survey article [Har02]). The first examples of groups of non-uniform exponential growth were constructed by J. Wilson [Wil04b, Wil04a] and later by L. Bartholdi [Bar03b], using self-similar groups.

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