An uncountable family of three-generated groups

Volodymyr Nekrashevych

November 22, 2005

Contents

1 Introduction **1**

2 Definition of the family **2**

3 Universal group of the family **5**
3.1 The construction . . . 5
3.2 Overgroup $\widetilde{G}$ and conjugacy classes of G_w . . . 5
3.3 Contraction . . . 8
3.4 Isomorphism classes . . . 9
3.5 Defining relations . . . 12

4 Overgroup $\widehat{G}$ and relation with holomorphic dynamics **14**
4.1 Definition of $\widehat{G}$ . . . 14
4.2 G_w as iterated monodromy groups . . . 16
4.3 The group $\widehat{G}$ as an iterated monodromy group . . . 19
4.4 Limit spaces of G and $\widehat{G}$ . . . 23

1 Introduction

We discuss here an amazing family of three-generated groups G_w parametrized by infinite binary sequences.

We prove that isomorphism classes in this family are countable and if two sequences w_1, w_2 are cofinal (i.e., differ only in a finite number of coordinates), then the groups G_{w_1} and G_{w_2} are isomorphic. In particular, this implies that the sets of sequences parametrizing groups of one isomorphism class are dense in the space X^ω of infinite binary sequences.

We also show that if w_1 and w_2 have a long common beginning, then a large ball of the Cayley graph of G_{w_1} is isomorphic to the ball of corresponding radius in the Cayley graph of G_{w_2}. This, together with density of the isomorphism classes implies, that one can not distinguish any two groups in our family just looking at a finite number of relations between the generators. For any finite set of relations in one group G_{w_1} there exists a generating set in any other group G_{w_2} for which the same set of relations are satisfied.

This is not the first example of an uncountable family of this sort (see, for example [2]), but our examples are very explicit and admit a rather deep investigation of their properties.

We also establish a connection of this family with iterations of the two dimensional holomorphic map

$$\begin{pmatrix} z \\ p \end{pmatrix} \mapsto \begin{pmatrix} 1 - \frac{z^2}{p^2} \\ 1 - \frac{1}{p^2} \end{pmatrix}.$$

And show also that the groups G_w are precisely the iterated monodromy groups of post-critically finite non-autonomous ("random") polynomial iterations.

2 Definition of the family

Consider three automorphisms g_0, g_1, g_2 of the rooted binary tree, which are defined by the recursions

$$g_0 = \sigma(1, g_2), \quad g_1 = (1, g_0), \quad g_2 = (1, g_1).$$

In other terms, these automorphisms are the states of the four-state automaton shown on Figure 1.

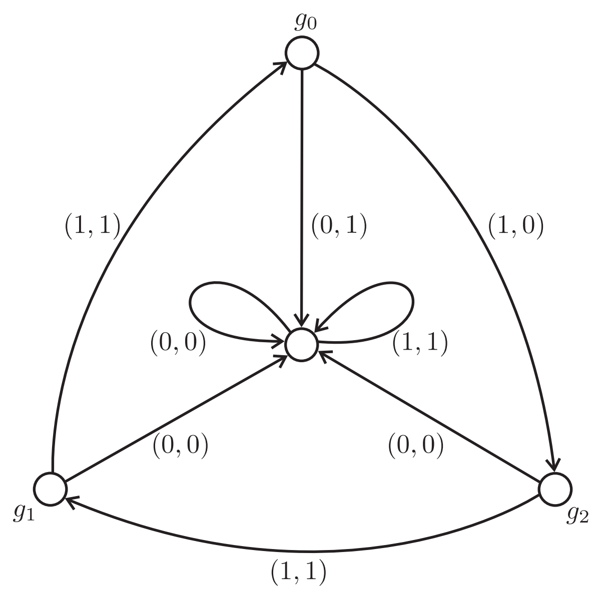

Figure 1: Automaton

The portraits of the automorphisms g_0, g_1, g_2 are shown on Figure 2.

The group generated by the automorphisms g_0, g_1, g_2 is called the *rabbit group*. The reasons for this name will be explained later in the article.

Two automorphisms of a rooted tree are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$ if and only if their *tree of orbits* are isomorphic (see [3]). Here the tree of orbits of an automorphism g is the tree of orbits of the cyclic group generated by g, where each vertex is labelled by the cardinality of the respective orbit.

Figure 3 shows the trees of orbits of the automorphisms g_0, g_1, g_2. In each of the trees the picture is periodic with period 3 along the right-most path of the tree. The period consists of two binary trees and one vertex having a single descendant.

Definition 2.1. An *r-triple* is a triple (h_0, h_1, h_2) of automorphisms of the tree X^*, conjugate in $\mathrm{Aut}(\mathsf{X}^*)$ (not necessary by the same conjugators) with the automorphisms g_0, g_1, g_2, respectively.

Let $w \in \mathsf{X}^\omega$ be an infinite binary sequence. Define the following elements $\alpha_w, \beta_w, \gamma_w$ of $\mathrm{Aut}(\mathsf{X}^*)$ recurrently

$$\alpha_w = \sigma\left(1, \gamma_{s(w)}\right),$$

$$\beta_w = \begin{cases} \left(1, \alpha_{s(w)}\right) & \text{if the first letter of } w \text{ is } 1, \\ \left(\alpha_{s(w)}, 1\right) & \text{if the first letter of } w \text{ is } 0, \end{cases}$$

$$\gamma_w = \left(1, \beta_{s(w)}\right).$$

Here $s(w)$ is the shift of the sequence w, i.e., the sequence obtained by deletion of its first letter.

For example, the generators g_0, g_1, g_2 of the rabbit group are equal to $\alpha_w, \beta_w, \gamma_w$, respectively, when $w = 111\ldots$.

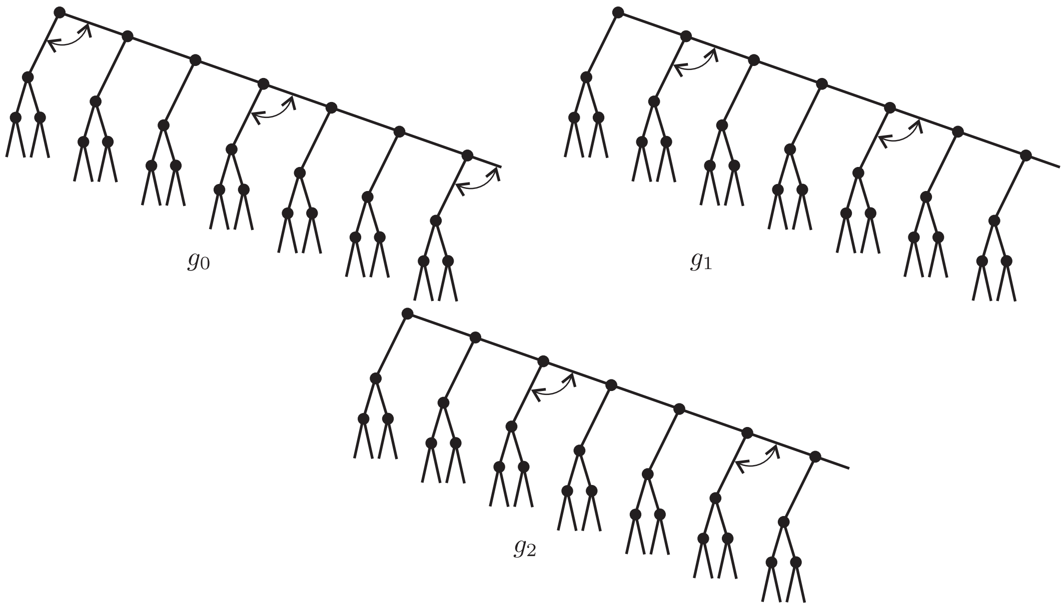

Figure 2: Portraits of g_0, g_1 and g_2

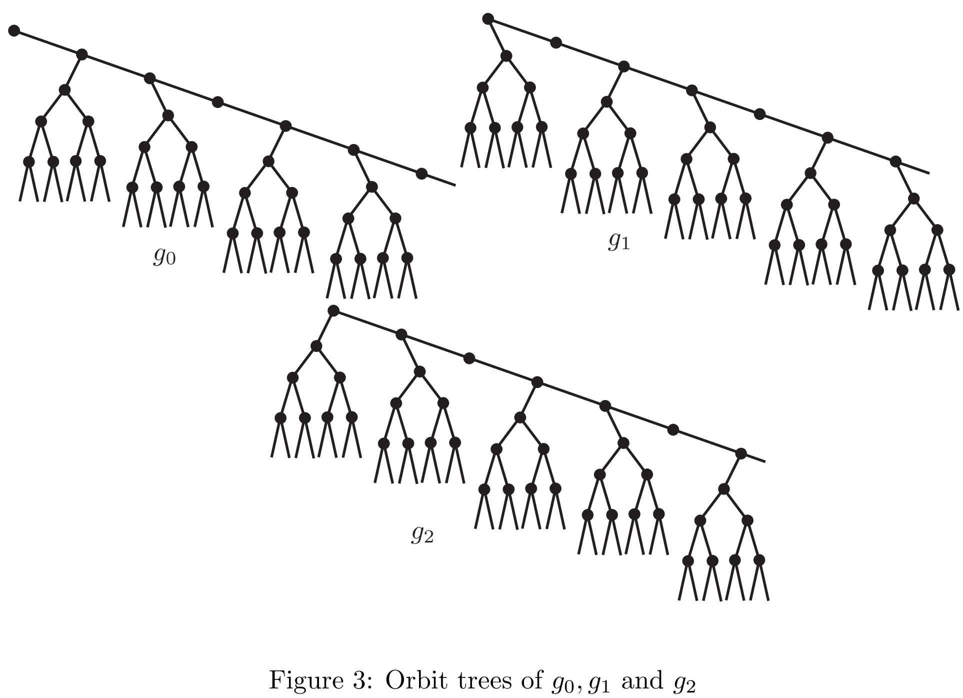

Figure 3: Orbit trees of g_0, g_1 and g_2

It is easy to see that $(\alpha_w, \beta_w, \gamma_w)$ is an r-triple, i.e., that each automorphism $\alpha_w, \beta_w, \gamma_w$ is conjugate in $\mathrm{Aut}(\mathsf{X}^*)$ to the automorphisms g_0, g_1 and g_2, respectively. One has just to check that the corresponding trees of orbits coincide.

Proposition 2.1. *Let (h_0, h_1, h_2) be an arbitrary r-triple. Then there exists a unique infinite sequence $w \in \mathsf{X}^\omega$ such that the triple (h_0, h_1, h_2) is simultaneously conjugate with the triple $(\alpha_w, \beta_w, \gamma_w)$, i.e., $\alpha_w = h^{-1}h_0h$, $\beta_w = h^{-1}h_1h$ and $\gamma_w = h^{-1}h_2h$ for some $h \in \mathrm{Aut}(\mathsf{X}^*)$.*

Proof. For $g \in \mathrm{Aut}(\mathsf{X}^*)$ denote by $g^{(1)}$ the automorphism (g, g). Inductively, $g^{(k)}$ is defined then by the condition that $g^{(k+1)} = \left(g^{(k)}\right)^{(1)}$. Then $g^{(k)}$ belongs to the kth level stabilizer and therefore the product $a_0a_1^{(1)}a_2^{(2)}\cdots$ converges in $\mathrm{Aut}(\mathsf{X}^*)$ for any sequence $a_0, a_1, \ldots$.

Suppose that we have a triple h_0, h_1, h_2 in which every h_i is conjugate with g_i. Then h_1 and h_2 are of the form $(1, a)$, or $(a, 1)$. Conjugating the triple h_0, h_1, h_2 by σ, if necessary, we may assume that h_2 is of the form $(1, h_1')$. Then h_1 is either of the form $(h_0', 1)$, or of the form $(1, h_0')$, where the automorphisms h_1' and h_0' are conjugate to g_1 and g_0, respectively. If h_1 and h_2 act non-trivially on the same subtrees of the first level, then the only possibility for the first digit of the sequence w, satisfying the conditions of the proposition, is 1. If they act on different subtrees, then it is 0. We see that the first digit of w is uniquely determined by the elements h_1 and h_2.

The element h_0 is conjugate to g_0, therefore it is of the form $\sigma\,(h_{00}, h_{01})$ and the product $h_{01}h_{00}$ is conjugate to g_2. Conjugating the triple by the element $(1, h_{00})$, we transform g_0 into the element

$$\left(1, h_{00}^{-1}\right)\sigma\,(h_{00}, h_{01})\,(1, h_{00}) = \sigma\,(1, h_{01}h_{00})\,.$$

Thus, we have proved that there exists a conjugator a_0 such that $h_0^{a_0} = \sigma(1, h_2'), h_1^{a_0} = (1, h_0')$ or $h_1^{a_0} = (h_0', 1)$ and $h_2^{a_0} = (1, h_1')$ for some elements h_0', h_1', h_2' conjugate to g_0, g_1, g_2, respectively.

A direct check shows that if b_0 is an element such that $h_0^{a_0b_0}$, $h_1^{a_0b_0}$ and $h_2^{a_0b_0}$ are also of the form $\sigma(1, h_2'')$, $(1, h_0'')$ or $(h_0'', 1)$, and $(1, h_1'')$, respectively, then $b_0 = (b, b) = b^{(1)}$ for some b, hence $h_i'' = {h_i'}^b$.

Consequently, we have proved that the triple (h_0', h_1', h_2') is defined uniquely up to a simultaneous conjugation.

We repeat now the same procedure for h_0', h_1' and h_2' and find the second possible letter of the sequence w and a conjugator a_1, transforming the automorphisms h_0', h_1' and h_2' into the nice form. Note that then $h_0^{a_0a_1^{(1)}} = \sigma\left(1, {h_2'}^{a_1}\right)$, $h_2^{a_0a_1^{(1)}} = \left(1, {h_2'}^{a_1}\right)$ and $h_1^{a_0a_1^{(1)}}$ is equal either to $\left({h_0'}^{a_1}, 1\right)$ or to $\left(1, {h_0'}^{a_1}\right)$, depending on the first letter of w.

We continue the procedure further (now with the states of ${h_i'}^{a_1}$), and at the end we will get all letters of the word w and a sequence $a_0, a_1, a_2, \ldots \in \mathrm{Aut}(\mathsf{X}^*)$ such that

$$h_0^{a_0a_1^{(1)}a_2^{(2)}a_3^{(3)}\cdots} = \alpha_w, \quad h_1^{a_0a_1^{(1)}a_2^{(2)}a_3^{(3)}\cdots} = \beta_w, \quad h_2^{a_0a_1^{(1)}a_2^{(2)}a_3^{(3)}\cdots} = \gamma_w.$$

□

The statement of the last proposition was probably noticed for the first time by R. Pink during his computations of the profinite iterated monodromy group of the "rabbit" polynomial (a private communication).

Let us denote by G_w the group, generated by α_w, β_w and γ_w.

Proposition 2.2. *For every word $w \in \mathsf{X}^\omega$ there exists at most a countable set of words $w' \in \mathsf{X}^\omega$ such that G_w is conjugate to $G_{w'}$ in $\mathrm{Aut}(\mathsf{X}^*)$.*

Proof. If G_w and $G_{w'}$ are conjugate, then there exists a generating set h_0, h_1, h_2 of G_w such that the triple (h_0, h_1, h_2) is conjugate to the triple $(\alpha_{w'}, \beta_{w'}, \gamma_{w'})$. The word w' is determined by the triple h_0, h_1, h_2 uniquely, by Proposition 2.1. But the number of possible generating sets $\{h_0, h_1, h_2\}$ of G_w is at most countable. □

3 Universal group of the family

3.1 The construction

Consider the following group $G = \langle \alpha, \beta, \gamma \rangle$ acting on the 4-regular tree $\mathsf{T} = \{1,2,3,4\}^*$:

$$\begin{aligned}\alpha &= \sigma\,(1,\gamma,1,\gamma)\\ \beta &= (\alpha,1,1,\alpha)\\ \gamma &= (1,\beta,1,\beta)\,,\end{aligned}$$

where $\sigma = (12)(34)$.

Let us identify the letters $1,2,3$ and 4 with the pairs $(0,0),(1,0),(0,1)$ and $(1,1)$, respectively. Then σ is the permutation $(0,y) \leftrightarrow (1,y)$. Hence, the elements of the group G leave in every word $(x_1,y_1)(x_2,y_2)\ldots(x_n,y_n)$ the second coordinates $y_1,y_2,\ldots,y_n$ unchanged.

Let us fix some infinite word $w = y_1y_2\ldots \in \mathsf{X}^\omega$. Then the set T_w of all vertices of the rooted tree $(\mathsf{X}\times\mathsf{X})^*$ of the form $(x_1,y_1)(x_2,y_2)\ldots(x_n,y_n)$ is a sub-tree isomorphic to the binary tree X^*. The isomorphism is the map

$$L_w : (x_1,y_1)(x_2,y_2)\ldots(x_n,y_n) \mapsto x_1x_2\ldots x_n.$$

The tree T_w is G-invariant. It follows directly from the recursions, that when we conjugate the action of α,β,γ on T_w by the isomorphism L_w, then we get the automorphisms $\alpha_w,\beta_w,\gamma_w$ of X^*.

We get thus the following description of G.

Proposition 3.1. *The restriction of the action of G onto the subtree T_w is conjugate with the action of G_w on the binary tree.*

The group G is isomorphic to the quotient $F\left/\bigcap_{w\in\mathsf{X}^\omega} N_w\right.$, where F is the free group generated by letters a,b,c and N_w is the kernel of the canonical epimorphism $F \to G_w$ mapping a,b and c to α_w, β_w and γ_w, respectively.

In other words, G is isomorphic to the group, generated by the diagonal elements

$$\alpha = (\alpha_w)_{w\in\mathsf{X}^\omega}, \beta = (\beta_w)_{w\in\mathsf{X}^\omega}, \gamma = (\gamma_w)_{w\in\mathsf{X}^\omega}$$

of the Cartesian product $\prod_{w\in\mathsf{X}^\omega} G_w$.

It is easy to prove by induction that every group G_w is level-transitive. Consequently, the quotient of the tree $(\mathsf{X}\times\mathsf{X})^*$ by the action of the group G is binary and is naturally identified with X^*. Two words $(x_1,y_1)\ldots(x_n,y_n)$ and $(x_1',y_1')\ldots(x_n',y_n')$ belong to one G-orbit if and only if $y_i = y_i'$. If w is an infinite path in the quotient-tree, then its preimage in $(\mathsf{X}\times\mathsf{X})^*$ is the binary tree T_w.

3.2 Overgroup $\widetilde{G}$ and conjugacy classes of G_w

Consider now the group generated by the following automorphisms

$$\begin{aligned}\alpha &= \sigma\,(1,\gamma,1,\gamma)\,, & a &= \pi\,(c,c,1,1)\,, & I_0 &= \left(I_2c\gamma^{-1}, I_2c, I_2\gamma^{-1}, I_2\right)\\ \beta &= (\alpha,1,1,\alpha)\,, & b &= (1,1,a,a)\,, & I_1 &= (I_0,I_0,I_0,I_0)\\ \gamma &= (1,\beta,1,\beta)\,, & c &= \left(1,\beta,b\beta^{-1},b\right), & I_2 &= (I_1,I_1,I_1,I_1)\,.\end{aligned}$$

where $\sigma = (12)(34)$ is the permutation $(0,x) \leftrightarrow (1,x)$ and $\pi = (13)(23)$ is the permutation $(x,0) \leftrightarrow (x,1)$.

Let us compute the conjugates of α,β,γ by a,b,c.

$$\begin{aligned}a\alpha a^{-1} &= \sigma\left(1,\gamma,1,c\gamma c^{-1}\right)\\ a\beta a^{-1} &= \left(1,\alpha,c\alpha c^{-1},1\right)\\ a\gamma a^{-1} &= \left(1,\beta,1,c\beta c^{-1}\right)\end{aligned}$$

$$\begin{aligned} b\alpha b^{-1} &= \sigma\left(1, \gamma, 1, a\gamma a^{-1}\right) \\ b\beta b^{-1} &= \left(\alpha, 1, 1, a\alpha a^{-1}\right) \\ b\gamma b^{-1} &= \left(1, \beta, 1, a\beta a^{-1}\right) \end{aligned}$$

$$\begin{aligned} c\alpha c^{-1} &= \sigma\left(\beta, \gamma\beta^{-1}, b\beta b^{-1}, b\beta^{-1}\gamma b^{-1}\right) \\ c\beta c^{-1} &= \left(\alpha, 1, 1, b\alpha b^{-1}\right) \\ c\gamma c^{-1} &= \left(1, \beta, 1, b\beta b^{-1}\right) \end{aligned}$$

We see from the recursions that $a\alpha a^{-1} = b\alpha b^{-1} = \alpha$, $b\beta b^{-1} = c\beta c^{-1} = \beta$, $a\gamma a^{-1} = c\gamma c^{-1} = \gamma$. The rest of conjugates are related by the recursion

$$\begin{aligned} c\alpha c^{-1} &= \sigma\left(\beta, \gamma\beta^{-1}, \beta, \beta^{-1} \cdot b\gamma b^{-1}\right) \\ a\beta a^{-1} &= \left(1, \alpha, c\alpha c^{-1}, 1\right) \\ b\gamma b^{-1} &= \left(1, \beta, 1, a\beta a^{-1}\right), \end{aligned}$$

which agree with the recursions

$$\begin{aligned} \gamma\alpha\gamma^{-1} &= \sigma\left(\beta, \gamma\beta^{-1}, \beta, \beta^{-1} \cdot \beta\gamma\beta^{-1}\right) \\ \alpha\beta\alpha^{-1} &= \left(1, \alpha, \gamma\alpha\gamma^{-1}, 1\right) \\ \beta\gamma\beta^{-1} &= \left(1, \beta, 1, \alpha\beta\alpha^{-1}\right). \end{aligned}$$

Consequently, we have the following relations

$$\begin{aligned} &a\alpha a^{-1} = \alpha, \quad a\beta a^{-1} = \alpha\beta\alpha^{-1}, \quad a\gamma a^{-1} = \gamma \\ &b\alpha b^{-1} = \alpha, \quad b\beta b^{-1} = \beta, \quad b\gamma b^{-1} = \beta\gamma\beta^{-1} \\ &c\alpha c^{-1} = \gamma\alpha\gamma^{-1}, \quad c\beta c^{-1} = \beta, \quad c\gamma c^{-1} = \gamma \end{aligned}$$

Let us consider now conjugation of α, β, γ by I_0, I_1, I_2. We get (using the known relations between $a, b, c, \alpha, \beta, \gamma$)

$$\begin{aligned} I_0\alpha I_0^{-1} &= \sigma\left(I_2\gamma I_2^{-1}, 1, I_2\gamma I_2^{-1}, 1\right) \\ I_0\beta I_0^{-1} &= \left(I_2\alpha I_2^{-1}, 1, 1, I_2\alpha I_2^{-1}\right) \\ I_0\gamma I_0^{-1} &= \left(1, I_2\beta I_2^{-1}, 1, I_2\beta I_2^{-1}\right). \end{aligned}$$

For $i = 1, 2$ we have

$$\begin{aligned} I_i\alpha I_i^{-1} &= \sigma\left(1, I_{i-1}\gamma I_{i-1}^{-1}, 1, I_{i-1}\gamma I_{i-1}^{-1}\right) \\ I_i\beta I_i^{-1} &= \left(I_{i-1}\alpha I_{i-1}^{-1}, 1, 1, I_{i-1}\alpha I_{i-1}^{-1}\right) \\ I_i\gamma I_i^{-1} &= \left(1, I_{i-1}\beta I_{i-1}^{-1}, 1, I_{i-1}\beta I_{i-1}^{-1}\right). \end{aligned}$$

All these relations imply

$$\begin{aligned} &I_0\alpha I_0^{-1} = \alpha^{-1}, \quad I_0\beta I_0^{-1} = \beta, \quad I_0\gamma I_0^{-1} = \gamma \\ &I_1\alpha I_1^{-1} = \alpha, \quad I_1\beta I_1^{-1} = \beta^{-1}, \quad I_1\gamma I_1^{-1} = \gamma \\ &I_2\alpha I_2^{-1} = \alpha, \quad I_2\beta I_2^{-1} = \beta, \quad I_2\gamma I_2^{-1} = \gamma^{-1} \end{aligned}$$

Definition 3.1. Let $\widetilde{G}$ be the group generated by the transformations $\alpha, \beta, \gamma, a, b, c, I_0, I_1, I_2$.

We have the following properties of the group $\widetilde{G}$.

Proposition 3.2. *The group $\widetilde{G}$ is level-transitive. The subgroup $G < \widetilde{G}$ is normal.*

Proof. The first statement follows from the fact that $\widetilde{G}$ is transitive on the first level of the tree and that it is recurrent (i.e., restriction of the stabilizer of a vertex onto the corresponding sub-tree is an epimorphism), which is checked directly.

Normality follows from the relations that we have proved above. □

We say that two sequences $w_1, w_2 \in \mathsf{X}^\omega$ are *cofinal* if they are of the form $w_1 = v_1 w, w_2 = v_2 w$ for some infinite word $w \in \mathsf{X}^\omega$ and two finite words $v_1, v_2 \in \mathsf{X}^*$ of equal length.

We say that $w_1, w_2 \in \mathsf{X}^\omega$ are *conjugate* if the groups G_{w_1} and G_{w_2} are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$.

Proposition 3.3. *Let $\widetilde{H}$ be the group acting on the binary tree and generated by the transformations*

$$g_0 = \sigma(1, g_2), g_1 = (1, g_0), g_2 = (1, g_1), r_0 = (g_2 r_2, r_2), r_1 = (r_0, r_0), r_2 = (r_1, r_1).$$

If $w_1, w_2 \in \mathsf{X}^\omega$ belong to one $\widetilde{H}$-orbit, then the groups G_{w_1} and G_{w_2} are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$ (and, in particular, are isomorphic).

In particular, the conjugacy classes are dense in X^ω and are unions of cofinality classes.

Proof. Consider the map from the tree $(\mathsf{X} \times \mathsf{X})^* \to \mathsf{X}^*$, induced by the map $(x, y) \mapsto y$. Then the action of $\widetilde{G}$ on $(\mathsf{X} \times \mathsf{X})^*$ agrees with this map (i.e., this map defines an imprimitivity system) so that the map induces an epimorphism of $\widetilde{G}$ onto the group of automorphisms of X^* equal to the action of $\widetilde{G}$ on the second coordinates of the letters. It is easy to see that G belongs to the kernel of this epimorphism, and that the generators a, b, c, I_0, I_1, I_2 are mapped to the generators $g_0^{-1}, g_1^{-1}, g_2^{-1}, r_0, r_1, r_2$ of $\widetilde{H}$, respectively.

If $w_2 = g(w_1)$ for some $g \in \widetilde{H}$, then for any preimage g' of g in $\widetilde{G}$ we have $g'(\mathsf{T}_{w_1}) = \mathsf{T}_{g(w_1)}$, where T_w, as before, denotes the subtree of $\mathsf{T} = (\mathsf{X} \times \mathsf{X})^*$ equal to union of the paths on whose second coordinates the word w is read.

This implies that the element g' conjugates the action of G on T_{w_1} with the action of G on T_{w_2}, which implies by Proposition 3.1 that G_{w_1} and G_{w_2} are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$.

It remains to prove that the orbits of the action of $\widetilde{H}$ on X^ω are unions of cofinality classes. But this follows from the fact that $\widetilde{H}$ is *recurrent*. □

Proposition 3.4. *The group $\widetilde{H}$ is contracting.*

Proof. The group $G_{111\ldots}$ generated by g_0, g_1, g_2 is contracting with the nucleus

$$\mathcal{N} = \left\{1, g_0, g_1, g_2, g_0 g_1^{-1}, g_1 g_2^{-1}, g_2 g_0^{-1}\right\}^{\pm 1}.$$

Relations

$$\begin{aligned} g_0 r_0 &= \sigma(g_2 r_2, g_2 r_2) & \qquad r_0 &= (g_2 r_2, r_2) \\ g_1 r_1 &= (r_0, r_0 g_0) & r_1 &= (r_0, r_0) \\ g_2 r_2 &= (r_1, g_1 r_1) & r_2 &= (r_1, r_1) \end{aligned}$$

imply that all the elements $r_0, r_1, r_2, g_0 r_0, g_1 r_1, g_2 r_2$ are of order two.

Relations

$$\begin{aligned} g_0 r_1 &= \sigma(r_0, g_2 r_0) & \qquad r_1 g_0 &= \sigma(r_0, r_0 g_2) \\ g_2 r_0 &= (g_2 r_2, g_1 r_2) & r_0 g_2 &= (g_2 r_2, r_2 g_1) \\ g_1 r_2 &= (r_1, g_0 r_1) & r_2 g_1 &= (r_1, r_1 g_0) \end{aligned}$$

show that $[g_0, r_1] = [g_2, r_0] = [g_1, r_2]$. Similarly

$$\begin{aligned} g_0 r_2 &= \sigma(r_1, g_2 r_1) & r_2 g_1 &= \sigma(r_1, r_1 g_2) \\ g_2 r_1 &= (r_0, g_1 r_0) & r_1 g_2 &= (r_0, r_0 g_1) \\ g_1 r_0 &= (g_2 r_2, g_0 r_2) & r_0 g_1 &= (g_2 r_2, r_2 g_0) \end{aligned}$$

implies that $[g_0, r_2] = [g_2, r_1] = [g_1, r_0]$. It is also checked directly that the elements r_0, r_1, r_2 commute.

Consequently, the elements r_0, r_1, r_2 generate the elementary abelian 2-group $C_2 \times C_2 \times C_2$ and the group $\widetilde{H}$ is a semi-direct product $(C_2 \times C_2 \times C_2) \ltimes \langle g_0, g_1, g_2 \rangle$. In particular, every element of $\widetilde{H}$ can be written in the form $r \cdot g$, where $g \in \langle g_0, g_1, g_2 \rangle$, and $r \in \langle r_0, r_1, r_2 \rangle$.

For any sufficiently long $v \in \mathsf{X}^*$ the restriction $g|_v$ belongs to the nucleus $\mathcal{N}$ of $\langle g_0, g_1, g_2 \rangle$. Then the restriction $(r \cdot g)|_v$ belongs to the set $Q \times \mathcal{N}$, where Q is the union of the sets of states of the elements of the group $\langle r_0, r_1, r_2 \rangle$. Direct computations show that Q is a finite set consisting of the following 20 elements

$$\begin{gathered} 1, \quad r_0 = (g_2 r_2, r_2), \quad r_1 = (r_0, r_0), \quad r_2 = (r_1, r_1), \\ r_0 r_1 = (g_2 r_0 r_2, r_0 r_2), \quad r_0 r_2 = (g_2 r_1 r_2, r_1 r_2), \quad r_1 r_2 = (r_0 r_1, r_0 r_1), \\ r_0 r_1 r_2 = (g_2 r_0 r_1 r_2, r_0 r_1 r_2), \\ g_0 r_0 = \sigma(g_2 r_2, g_2 r_2), \quad g_1 r_1 = (r_0, g_0 r_0), \quad g_2 r_2 = (r_1, g_1 r_1), \\ g_0 r_0 r_1 = \sigma(g_2 r_0 r_2, g_2 r_0 r_2), \quad g_0 r_0 r_2 = \sigma(g_2 r_1 r_2, g_2 r_1 r_2), \\ g_1 r_1 r_2 = (r_0 r_1, g_0 r_0 r_1), \quad g_1 r_0 r_1 = (g_2 r_0 r_2, g_0 r_0 r_2), \\ g_2 r_0 r_2 = (g_2 r_1 r_2, g_1 r_1 r_2), \quad g_2 r_1 r_2 = (r_0 r_1, g_1 r_0 r_1), \\ g_0 r_0 r_1 r_2 = \sigma(g_2 r_0 r_1 r_2, g_2 r_0 r_1 r_2), \\ g_1 r_0 r_1 r_2 = (r_0 r_1 r_2, g_0 r_0 r_1 r_2), \quad g_2 r_0 r_1 r_2 = (r_0 r_1 r_2, g_1 r_0 r_1 r_2). \end{gathered}$$

Consequently, the group $\widetilde{H}$ is contracting with the nucleus a subset of $Q \cdot \mathcal{N}$. □

It is easy to prove that the limit space $\mathcal{X}_{\widetilde{H}}$ is homeomorphic to the limit space $\mathcal{X}_{\langle g_0, g_1, g_2 \rangle}$. The proof is the same as the proof of Theorem 3.7.1 (2) in [5], only instead of injectivity of ϕ one has to use the fact that the virtual associated endomorphism preserves the adjacency classes of $\widetilde{H}$ by $\langle g_0, g_1, g_2 \rangle$.

Problem 1. Describe the limit space of $\widetilde{H}$. The last fact about $\mathcal{X}_{\widetilde{H}}$ implies that it is an 8-to-one orbispace quotient of the "Douady rabbit".

3.3 Contraction

Let F be the free group freely generated by the set of symbols α, β, γ. Define the following two homomorphisms (wreath recursions) ψ_0 and $\psi_1 : F \to \mathfrak{S}(\mathsf{X}) \wr F$:

$$\psi_i(\alpha) = \sigma(1, \gamma), \quad \psi_i(\beta) = \begin{cases} (\alpha, 1) & \text{if } i = 0, \\ (1, \alpha) & \text{if } i = 1, \end{cases} \quad \psi_i(\gamma) = (1, \beta).$$

These two wreath recursions define the respective permutational F-bimodules Ψ_0 and Ψ_1.

The direct sum of the bimodules $\Psi = \Psi_0 \oplus \Psi_1$ is precisely the bimodule associated with the self-similar group G.

For every binary word $w = x_1 x_2 \ldots x_n \in \mathsf{X}^*$ consider the tensor product of the F-bimodules $\Psi_{x_1} \otimes \Psi_{x_2} \otimes \cdots \otimes \Psi_{x_n} = \Psi_w$ and let $\psi_w : F \to \underbrace{\mathfrak{S}(\mathsf{X}) \wr \mathfrak{S}(\mathsf{X}) \wr \cdots \wr \mathfrak{S}(\mathsf{X})}_{n \text{ times}} \wr F$ be the respective wreath recursion.

Proposition 3.5. *Denote by $\mathcal{E}_w$ the kernel of the homomorphism ψ_w. Then $\mathcal{E}_{x_1\ldots x_{n-1}} \leq \mathcal{E}_{x_1\ldots x_{n-1}x_n}$ and the union $\mathcal{E}_{x_1x_2\ldots} = \bigcup_{n\geq 1} \mathcal{E}_{x_1\ldots x_n}$ is equal to the kernel of the natural epimorphism $F \to G_w$, carrying α, β, γ to $\alpha_{x_1x_2\ldots}, \beta_{x_1x_2\ldots}, \gamma_{x_1x_2\ldots}$, respectively.*

Proof. The F-bimodule Ψ is contracting, since the overgroup $\widehat{G}$ of G is the iterated monodromy group of a hyperbolic rational function F (see the next sections).

A more direct computation shows that the set

$$\mathcal{N} = \{1, \alpha^{\pm 1}, \beta^{\pm 1}, \gamma^{\pm 1}, (\beta\alpha)^{\pm 1}, (\alpha\gamma)^{\pm 1}, (\gamma\beta)^{\pm 1}, (\beta\alpha^{-1})^{\pm 1}, (\alpha\gamma^{-1})^{\pm 1}, (\gamma\beta^{-1})^{\pm 1}, (\gamma\beta\alpha^{-1})^{\pm 1}, (\beta\alpha\gamma^{-1})^{\pm 1}, (\alpha\gamma\beta^{-1})^{\pm}\}$$

is the nucleus of the free group F with respect to the bimodule Ψ. We see that the nucleus does not contain elements, which are trivial in G. This implies, that the kernel of the canonical epimorphism $F \to G$ is equal to the union $\mathcal{E}$ of the kernels of the wreath recursions associated with the F-bimodule Ψ.

This implies the statement of the proposition, due to Proposition 3.1 and [5, Proposition 2.13.2] □

Corollary 3.6. *Let $R \subset \mathcal{E}_w$ be a finite subset of the kernel of the canonical epimorphism $F \to G_w$ and let $U \subset F \setminus \mathcal{E}_w$ be a finite subset of its complement. Then there exists a neighborhood W of the word w in X^ω such that $R \subset \mathcal{E}_{w'}$ and $U \subset F \setminus \mathcal{E}_{w'}$ for all $w' \in W$.*

Proof. The previous proposition shows that any finite set of relations and inequalities between the generators of G_w can be detected from the wreath recursion ψ_v, where v is a beginning of w. But then the same relations and inequalities hold in all groups $G_{w'}$, where w' starts with v. □

Proposition 3.7. *Let $w_1, w_2 \in \{0,1\}^\omega$. For any finite set of relations and inequalities between the generators $\alpha_{w_1}, \beta_{w_1}$ and γ_{w_1} there exist generators h_0, h_1 and h_2 of the group G_{w_2} such that the same set of relations and inequalities hold for the generators h_0, h_1 and h_2 in the group G_{w_2}.*

Proof. A direct corollary of Proposition 3.3 and Corollary 3.6. □

3.4 Isomorphism classes

Lemma 3.8. *For every $w \in G_w$ the elements $\alpha_w, \beta_w, \gamma_w$ freely generate the abelianization*

$$G_w/[G_w, G_w] \cong \mathbb{Z}^3.$$

Proof. Let F_w be the free group generated by $\alpha_w, \beta_w, \gamma_w$. Let f_w be the homomorphism $f_w : F_w \to \mathbb{Z}^3 = \langle e_1, e_2, e_3\rangle$ extending the map $\alpha_w \mapsto e_1$, $\beta_w \mapsto e_2$, $\gamma_w \mapsto e_3$. Suppose the proposition false, and let g be an element of F_w representing a trivial element of G_w, and whose image $f_w(g)$ in $\mathbb{Z}^3$ is non-zero.

If we apply the wreath recursion to g, then we get $g = (g_0, g_1)$. In particular, g can be written as a word in the generators of the first level stablizer

$$\begin{aligned} \alpha_w^2 &= \left(\gamma_{s(w)}, \gamma_{s(w)}\right) \\ \beta_w &= \left(1, \alpha_{s(w)}\right), \text{ or } \left(\alpha_{s(w)}, 1\right) \\ \gamma_w &= \left(1, \beta_{s(w)}\right) \\ \alpha_w^{-1}\beta_w\alpha_w &= \left(\alpha_{s(w)}, 1\right), \text{ or } \left(1, \gamma_{s(w)}^{-1}\alpha_{s(w)}\gamma_{s(w)}\right) \\ \alpha_w^{-1}\gamma_w\alpha_w &= \left(\beta_{s(w)}, 1\right). \end{aligned}$$

The recurrent formulae show that $f_{s(w)}(g_0) + f_{s(w)}(g_1) = T(f_w(g))$, where $T : \mathbb{Z}^3 \to \mathbb{Z}^3$ is the automorphism

$$e_1 \mapsto e_3, \quad e_2 \mapsto e_1, \quad e_3 \mapsto e_2.$$

Consequently, either $f(g_0)$ or $f(g_1)$ is non-trivial. Continuing further, we get by induction that for every level $\{0,1\}^n$ there exists $v \in \{0,1\}^n$ such that $g|_v$ is non-trivial in $\mathbb{Z}^2$, but obviously is trivial in $G_{s^{|v|}(w)}$.

But the group G is contracting with the nucleus given in the proof of Proposition 3.5, whence we get a contradiction. □

Recall (see [1] and [5, Definition 1.2.4]) that a group $G \leq \mathrm{Aut}(\mathsf{X}^*)$ is said to be *weakly branch* if for every vertex $v \in \mathsf{X}^*$ there exists a non-trivial element $g \in G$ acting trivially on all the words not starting with v.

Proposition 3.9. *The group G_w is weakly branch for every $w \in \mathsf{X}^\omega$. Moreover, the image of $G'_w = [G_w, G_w]$ under the wreath recursion contains the direct product $G'_{s(w)} \times G'_{s(w)}$, which is* geometric*, i.e., the factors act on the respective subtrees of the first level.*

Proof. Let us denote by $\pi_w : G_w \to \mathbb{Z}^3$ the abelianization map $\alpha_w \mapsto (1,0,0)$, $\beta_w \mapsto (0,1,0)$ and $\gamma_w \mapsto (0,0,1)$.

It follows from the recursion that the stabilizer of the first level in G_w is generated by the elements $(\alpha_{s(w)}, 1)$, $(1, \alpha_{s(w)})$, $(1, \beta_{s(w)})$, $(\beta_{s(w)}, 1)$ and $(\gamma_{s(w)}, \gamma_{s(w)})$. Consequently, for $g_0, g_1 \in G_{s(w)}$, the automorphism (g_0, g_1) of the tree belongs to G_w if and only if the third coordinates of $\pi_{s(w)}(g_0)$ and $\pi_{s(w)}(g_1)$ are equal.

It follows from the recurrent definition of the generators of G_w that the map $\pi_w((g_0, g_1)) \mapsto \pi_{s(w)}(g_0) + \pi_{s(w)}(g_1)$ is a well defined isomorphism (it is actually the main idea of the proof of Lemma 3.8).

Consequently, if (g_0, g_1) belongs to $G'_{s(w)} \times G'_{s(w)}$, then $\pi_{s(w)}(g_0) = \pi_s(w)(g_1) = \vec{0}$, hence it belongs to G'_w. □

Proposition 3.10. *Define $G_w^{2^n}$ inductively as the group generated by the squares of the group $G_w^{2^{n-1}}$ and setting $G_w^{2^0} = G_w$. Then for every $n \geq 0$ the group $G_w^{2^n}$ belongs to the nth level stabilizer and acts level-transitively on each of the subtrees with roots on the nth level.*

Proof. It is easy to prove by induction that $G_w^{2^n}$ belongs to the nth level stabilizer.

The group G_w^2 contains $(\gamma_{s(w)}, \gamma_{s(w)}) = \alpha_w^2$, $(\beta_{s(w)}, \beta_{s(w)}) = \alpha_w^{-2}(\alpha_w\gamma_w)^2$ and $(\alpha_{s(w)}, \alpha_{s(w)})$, which is equal either to $\alpha_w^{-2}(\alpha_w\beta_w)^2$ or to $\alpha_w^2(\alpha^{-1}\beta_w)^2$ (depending on the first letter of w).

We conclude that the image of G_w^2 in $G_{s(w)} \times G_{s(w)}$ contains the diagonal of the direct product, hence, by induction the image of $G_w^{2^n}$ in the 2^nth direct power of $G_{s^n(w)}$ also contains the diagonal. This implies that the action of $G_w^{2^n}$ is level-transitive on the sub-trees of the nth level, since the groups G_w are level-transitive. □

Theorem 3.11. *Two groups G_{w_1} and G_{w_2} are isomorphic as abstract groups if and only if they are conjugate in* $\mathrm{Aut}(\mathsf{X}^*)$ *and automorphism group of G_{w_i} coincides with the normalizer of G_{w_i} in* $\mathrm{Aut}(\mathsf{X}^*)$.

Proof. A direct corollary of Propositions 3.9, 3.10 and a result of [4]. (See also [5, Proposition 2.10.7]). □

Let us call two sequences $w_1, w_2 \in \mathsf{X}^\omega$ *isomorphic* if the groups G_{w_1} and G_{w_2} are isomorphic. The last theorem says that two sequences are isomorphic if and only if they are conjugate. We get now from Proposition 3.3.

Corollary 3.12. *The isomorphism classes are countable, dense in X^ω and are unions of cofinality classes.*

Moreover, the isomorphism relation is completely invariant under the shift, i.e., the sequences w_1, w_2 are isomorphic if and only if the shifted sequences $s(w_1)$ and $s(w_2)$ are isomorphic. In one direction it is the last corollary, in the other direction it is the obvious fact that the conjugacy class of G_w uniquely determines the conjugacy class of $G_{s(w)}$, since $G_{s(w)}$ is the group generated by the restrictions of the elements of G_w onto the subtrees of the first level.

We have reduced isomorphism problem to answering the question when two groups from the family G_w are conjugate.

Problem 2. Describe the isomorphism classes of G_w. Is it true that two groups G_{w_1} and G_{w_2} are isomorphic if and only if w_1, w_2 belong to one $\widetilde{H}$-orbit? This seems to be the most reasonable conjecture. If it is not true, then probably isomorphism classes have no simple description.

The rest of this subsection is a collection of results (the main is Proposition 3.16) supporting the conjecture from Problem 2, or at least giving some more information on the isomorphism classes of the groups G_w.

A group G_{w_1} is isomorphic to G_w if and only if there exists a generating set g_0, g_1, g_2 of G_w, which is simultaneously conjugate to $\alpha_{w_1}, \beta_{w_1}, \gamma_{w_1}$ in $\mathrm{Aut}(\mathsf{X}^*)$.

Definition 3.2. We will denote by $\pi_\alpha, \pi_\beta, \pi_\gamma$ the homomorphisms from G_w (for any w) onto $\mathbb{Z}$, which maps all standard generators to 0, except for $\alpha_w, \beta_w, \gamma_w$, respectively, which are mapped to 1.

Lemma 3.13. *Let f_0, f_1, f_2 be an r-triple of elements of G_w. Then all the numbers $\pi_\beta(f_0)$, $\pi_\gamma(f_0)$, $\pi_\alpha(f_1)$, $\pi_\gamma(f_1)$, $\pi_\alpha(f_2)$, $\pi_\beta(f_2)$ are equal to zero.*

Proof. Then f_1 is of the form $(1, h_0)$ or $(h_0, 1)$, f_2 is of the form $(1, h_1)$ or $(h_1, 1)$ and f_0 is of the form $\sigma(f_{00}, f_{01})$ for some $h_0, h_1, f_{00}, f_{01} \in G_{s(w)}$ such that $(h_0, h_1, h_2 = f_{00}f_{01})$ is also an r-triple.

We know (see the proof of Proposition 3.9) that (x, y) belongs to G_w if and only if $x, y \in G_{s(w)}$ are such that $\pi_\gamma(x) = \pi_\gamma(y)$. It follows that $\pi_\gamma(h_0) = \pi_\gamma(h_1) = 0$.

Consequently, $\pi_\alpha(f_1) = \pi_\alpha(f_2) = 0$. This equality is true for every r-triple f_0, f_1, f_2, hence it is also true for h_1, h_2, which implies that $\pi_\beta(f_0) = \pi_\beta(f_2) = 0$. □

As a direct corollary we get.

Lemma 3.14. *If (f_0, f_1, f_2) is an r-triple of generators of G_w, then $\pi(f_0) = (\pm 1, 0, 0)$, $\pi(f_1) = (0, \pm 1, 0)$ and $\pi(f_2) = (0, 0, \pm 1)$.*

Lemma 3.15. *Suppose that (f_0, f_1, f_2) is an r-triple in G_w.*

Then

1. *f_0 is conjugate to α_w in G_w if and only if $\pi(f_0) = (1, 0, 0)$,*
2. *f_1 is conjugate to β_w in G_w if and only if $\pi(f_1) = (0, 1, 0)$,*
3. *f_2 is conjugate to γ_w in G_w if and only if $\pi(f_2) = (0, 0, 1)$.*

Proof. The statement is obvious in the "only if" direction. Let us prove the "if" direction simultaneously for f_0, f_1 and f_2.

We have $f_0 = \sigma(f_{00}, f_{01})$, $f_1 = (h_0, 1)$ or $(1, h_0)$ and $f_2 = (h_1, 1)$ or $(1, h_1)$ for some $f_{00}, f_{01}, h_0, h_1 \in G_{s(w)}$ such that $h_0, h_1, h_2 = f_{00}f_{01}$ is an r-triple in $G_{s(w)}$ with the same images under π as f_0, f_1, f_2.

We have $\pi_\gamma(f_{00}) + \pi_\gamma(f_{01}) = \pi_\gamma(h_2) = 1$. On the other hand, the element

$$f_0\alpha_w^{-1} = \sigma(f_{00}, f_{01})(1, \gamma_{s(w)}^{-1})\sigma = (f_{01}\gamma_{s(w)}^{-1}, f_{00})$$

belongs to G_w, hence $\pi_\gamma(f_{01}) - 1 = \pi_\gamma(f_{01}\gamma_{s(w)}^{-1}) = \pi_\gamma(f_{00})$.

Consequently, we get $\pi_\gamma(f_{00}) = 0$ and $\pi_\gamma(f_{01}) = 1$. It follows that $(f_{00}, 1) \in G_w$.

We get $(f_{00}, 1)f_0(f_{00}, 1)^{-1} = \sigma(1, f_{00}f_{01})$.

Consequently, if we prove our statement for h_0, h_1 or h_2 in $G_{s(w)}$, then we prove it for f_1, f_2 and f_0, respectively, in G_w.

In particular, it means that if we find a counterexample to one of the statements, then we find a counterexample to all three statements.

We obviously have inequalities $l(h_0) \le l(f_1)$, $l(h_1) \le l(f_1)$ and $l(h_2) \le l(f_0)$, hence the shortest counterexamples to each of the three statements have the same length.

Suppose now that f_2 is a shortest counterexample (for the third statement of the proposition).

Suppose that the first letter of w is 0. Then $f_2 \in \langle \beta_w, \gamma_w \rangle$, if it is of the form $(1, h_1)$, and then $h_1 \in \langle \alpha_{s(w)}, \beta_{s(w)} \rangle$, since in all the other case the length of h_1 will be less than the length of f_2. Similarly, $f_2 \in \langle \alpha_w \beta_w \alpha_w^{-1}, \alpha_w \gamma_w \alpha^{-1} \rangle$ if $f_2 = (1, h_1)$, and then $h_1 \in \langle \gamma_{s(w)} \alpha_{s(w)} \gamma_{s(w)}^{-1}, \gamma_{s(w)} \beta_{s(w)} \gamma_{s(w)}^{-1} \rangle$.

If the first letter of w is 1, then $f_2 \in \langle \alpha_w^{-1} \beta_w \alpha_w, \gamma_w \rangle$ and $h_1 \in \langle \gamma_{s(w)}^{-1} \alpha_{s(w)} \gamma_{s(w)}, \beta_{s(w)} \rangle$ if $f_2 = (1, h_1)$. If $f_2 = (h_1, 1)$, then $f_2 \in \langle \beta_w, \alpha_w \gamma_w \alpha_w^{-1} \rangle$ and $h_1 \in \langle \alpha_{s(w)}, \gamma_{s(w)} \beta_{s(w)} \gamma_{s(w)}^{-1} \rangle$.

But h_1 is also a minimal counterexample for the second statement of the proposition, so the conditions on f_2 are also true for h_1. Putting all conditions together, we get that h_1 has to be of the form $\beta_{s(w)}^n$ or $\gamma_{s(w)} \beta_{s(w)}^n \gamma_{s(w)}^{-1}$. But the condition on the image of h_1 under π implies that $n = 1$, i.e., that h_1 is not a counterexample. □

We have thus proved the following description of the r-triples of generators.

Proposition 3.16. *If (f_0, f_1, f_2) is an r-triple of generators of G_w, then there exist $x, y, z \in G_w$ such that $f_0 = \alpha_w^{\pm x}, f_1 = \beta_w^{\pm y}, f_2 = \gamma_w^{\pm z}$.*

3.5 Defining relations

Let F be the free group generated by the symbols α, β, γ. Consider the following endomorphisms of F.

$$\begin{aligned} \varphi_0(\alpha) &= \alpha\beta\alpha^{-1}, & \varphi_1(\alpha) &= \beta, \\ \varphi_0(\beta) &= \gamma, & \varphi_1(\beta) &= \gamma, \\ \varphi_0(\gamma) &= \alpha^2, & \varphi_1(\gamma) &= \alpha^2. \end{aligned}$$

Each group G_w is a homomorphic image of the group F under the *canonical* epimorphism $\alpha \mapsto \alpha_w, \beta \mapsto \beta_w, \gamma \mapsto \gamma_w$. Our aim is to find generators of the kernel as a normal subgroup of F, i.e., a presentation for G_w.

Lemma 3.17. *The homomorphism φ_i induces, for every $w \in \mathsf{X}^\omega$, an injective homomorphism $G_w \to G_{iw}$ such that $\varphi_i(g) = \left(\gamma_w^{\pi_\gamma(g)}, g\right)$.*

For definition of the homomorphism π_γ see Definition 3.2.

Proof. Let us compute the decompositions of the images of generators of G_w under $\psi_i \circ \varphi_i$, where ψ_i is the corresponding wreath recursion (see 3.3).

For $i = 0$ we have

$$\begin{aligned} \psi_0 \circ \varphi_0(\alpha) &= \psi_0(\alpha\beta\alpha^{-1}) = (1, \alpha) \\ \psi_0 \circ \varphi_0(\beta) &= \psi(\gamma) = (1, \beta) \\ \psi_0 \circ \varphi_0(\gamma) &= \psi(\alpha^2) = (\gamma, \gamma). \end{aligned}$$

For $i = 1$ we have

$$\begin{aligned} \psi_1 \circ \varphi_1(\alpha) &= \psi_1(\beta) = (1, \alpha) \\ \psi_1 \circ \varphi_1(\beta) &= \psi_1(\gamma) = (1, \beta) \\ \psi_1 \circ \varphi_1(\alpha) &= \psi_1(\alpha^2) = (\gamma, \gamma). \end{aligned}$$

It follows that for every $g \in G_w$ the element $\varphi_i(g) \in G_{iw}$ is well defined and is equal to $\left(\gamma_w^{\pi_\gamma(g)}, g\right)$, and hence is a well defined monomorphism. □

For a definition and properties of the kernels $\mathcal{E}_{x_1 x_2 \ldots x_n}$ see Proposition 3.5.

Lemma 3.18. *The subgroups $\mathcal{E}_0$ and $\mathcal{E}_1$ are equal to the subgroups $\mathcal{R}_0$, $\mathcal{R}_1$, given by*

$$\mathcal{R}_0 = \left\langle \left[\beta^{\alpha^{2n}}, \gamma\right], \left[\beta^{\alpha^{2n+1}}, \beta\right], \left[\gamma^{\alpha^{2n+1}}, \gamma\right] \ : \ n \in \mathbb{Z} \right\rangle^F$$
$$\mathcal{R}_1 = \left\langle \left[\beta^{\alpha^{2n+1}}, \gamma\right], \left[\beta^{\alpha^{2n+1}}, \beta\right], \left[\gamma^{\alpha^{2n+1}}, \gamma\right] \ : \ n \in \mathbb{Z} \right\rangle^F .$$

Proof. For $i = 0$ we have

$$\psi_0\left(\alpha^n\beta\alpha^{-n}\right) = \begin{cases} \left(\gamma^{n/2}\alpha\gamma^{-n/2}, 1\right) & \text{if } n \text{ is even,} \\ \left(1, \gamma^{(n-1)/2}\alpha\gamma^{-(n-1)/2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

For $i = 1$ we have

$$\psi_1\left(\alpha^n\beta\alpha^{-n}\right) = \begin{cases} \left(1, \gamma^{n/2}\alpha\gamma^{-n/2}\right) & \text{if } n \text{ is even,} \\ \left(\gamma^{(n+1)/2}\alpha\gamma^{-(n+1)/2}, 1\right) & \text{if } n \text{ is odd.} \end{cases}$$

For any i

$$\psi_i\left(\alpha^n\gamma\alpha^{-n}\right) = \begin{cases} \left(1, \gamma^{n/2}\beta\gamma^{-n/2}\right) & \text{if } n \text{ is even,} \\ \left(\gamma^{(n+1)/2}\beta\gamma^{-(n+1)/2}, 1\right) & \text{if } n \text{ is odd.} \end{cases}$$

We see that $\mathcal{R}_i \leq \mathcal{E}_i$. Let us prove the converse inclusions.

Every element $g \in \mathcal{E}_i$ can be written it in the form $\alpha^n h_1 h_2 \cdots h_m$, where h_i are powers of conjugates of β and γ by powers of α. By Lemma 3.8 we have $n = 0$.

Since $g \in \mathcal{E}_i$, we must have $\psi_i(g) = 1$ in $\mathfrak{S}(\mathsf{X}) \wr F$, i.e., the products of the elements $\psi_i(h_i)$ is trivial in $F \times F$. Modulo the relators from $\mathcal{R}_i$ we can separate the factors h_j, which have trivial first coordinate of $\psi_i(h_j)$, from the factors having trivial second coordinate of $\psi_i(h_j)$. The products of the non-trivial coordinates are equal to 1 in F. But this implies that modulo $\mathcal{R}_i$ the product $h_1 h_2 \cdots h_m$ is also trivial. □

Lemma 3.19. *For every $x_1 \ldots x_n$ we have*

$$\mathcal{E}_{x_1 x_2 \ldots x_n} = \mathcal{E}_{x_1} \cdot \varphi_{x_1}(\mathcal{E}_{x_2 \ldots x_n}) \cdot \varphi_{x_1}(\mathcal{E}_{x_2 \ldots x_n})^\alpha$$

Proof. Consider $g \in \mathcal{E}_{x_1 x_2 \ldots x_n}$. We can write it in the form $\alpha^n h_1 h_2 \cdots h_m$, where h_i are powers of conjugates of β and γ by powers of α. By Lemma 3.8 we have $n = 0$.

Let (g_0, g_1) be the image of g under the wreath recursion. Then, by definition of $\mathcal{E}_{x_1 x_2 \ldots x_n}$, $g_i \in \mathcal{E}_{x_2 \ldots x_n}$ and by Lemma 3.17, $(1, g_0) = \varphi_{x_1}(g_0)$, $(g_1, 1) = \alpha^{-1}\varphi_{x_1}(g_1)\alpha$, hence we see that $g \in \mathcal{E}_{x_1} \cdot \varphi_{x_1}(\mathcal{E}_{x_2 \ldots x_n}) \cdot \alpha^{-1}\varphi_{x_1}(\mathcal{E}_{x_2 \ldots x_n})\alpha$. □

We get finally the following description of the defining relations for the groups G_w.

Theorem 3.20. *Let*

$$R_0 = \left\{ \left[\beta^{\alpha^{2n}}, \gamma\right], \left[\beta^{\alpha^{2n+1}}, \beta\right], \left[\gamma^{\alpha^{2n+1}}, \gamma\right] \ : \ n \in \mathbb{Z} \right\}$$

and

$$R_1 = \left\{ \left[\beta^{\alpha^{2n+1}}, \gamma\right], \left[\beta^{\alpha^{2n+1}}, \beta\right], \left[\gamma^{\alpha^{2n+1}}, \gamma\right] \ : \ n \in \mathbb{Z} \right\},$$

and let φ_i be the endomorphisms

$$\begin{aligned} \varphi_0(\alpha) &= \alpha\beta\alpha^{-1}, & \varphi_1(\alpha) &= \beta, \\ \varphi_0(\beta) &= \gamma, & \varphi_1(\beta) &= \gamma, \\ \varphi_0(\gamma) &= \alpha^2, & \varphi_1(\gamma) &= \alpha^2. \end{aligned}$$

Then for every sequence $w = x_1 x_2 \ldots \in \mathsf{X}^\omega$ the set

$$\bigcup_{n=1}^{\infty} \varphi_{x_1} \circ \varphi_{x_2} \circ \cdots \circ \varphi_{x_{n-1}}(R_{x_n})$$

is a set of defining relations of the group G_w.

Let g_0, g_1, g_2 be the automorphisms of the free group F, given by

$$\begin{array}{lll} g_0(\alpha) = \alpha, & g_1(\alpha) = \alpha, & g_2(\alpha) = \gamma^{-1}\alpha\gamma \\ g_0(\beta) = \alpha^{-1}\beta\alpha, & g_1(\beta) = \beta, & g_2(\beta) = \beta \\ g_0(\gamma) = \gamma, & g_1(\gamma) = \beta^{-1}\gamma\beta, & g_2(\gamma) = \gamma, \end{array}$$

(compare with Subsection 3.2), and let r_0, r_1, r_2 be the automorphisms of F, inverting the generators α, β, γ, respectively.

The following proposition gives us a (maybe more explicit) alternative proof of Propositions 3.3 and 3.7.

Proposition 3.21. *Let $\widetilde{H}_1$ be the group generated by the automorphisms g_i, r_i of F. Then the group $\widetilde{H}$ is a quotient of $\widetilde{H}_1$ with the epimorphism given by the tautological map on the generators $g_i \mapsto g_i, r_i \mapsto r_i$. Consider the obtained action of $\widetilde{H}_1$ on X^*.*

If for $g \in \widetilde{H}_1$ we have $g(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n$, then the normal closure of

$$g\left(\varphi_{x_1} \circ \varphi_{x_2} \circ \cdots \circ \varphi_{x_{n-1}}(R_{x_n})\right)$$

in F coincides with the normal closure of

$$\varphi_{y_1} \circ \varphi_{y_2} \circ \cdots \circ \varphi_{y_{n-1}}(R_{y_n}).$$

Proof. We have the following equalities, which are checked directly

$$g_0 \circ \varphi_0 = \varphi_1, \quad g_0 \circ \varphi_1 = \varphi_0 \circ g_2.$$

and

$$g_1 \circ \varphi_1 = \varphi_1 \circ g_0, \quad g_2 \circ \varphi_1 = \varphi_1 \circ g_1, \quad g_0^2 \circ \varphi_1 = \varphi_1 \circ g_2.$$

We also obviously have

$$r_0 \circ \phi_1 = \phi_1 \circ r_2, \quad r_1 \circ \phi_1 = \phi_1 \circ r_0, \quad r_2 \circ \phi_1 = \phi_1 \circ r_1.$$

These equalities coincide with the recurrent definitions of the generators of $\widetilde{H}$ (with the self-similarity bimodule associated to its action on X^*).

Note, finally, that $g_0(R_1) = R_0$, $g_0(R_1) = R_0$ and that

$$\langle g_j(R_i) \rangle^F = \langle R_i \rangle^F$$

for all $i = 1, 0$ and $j = 1, 2$. □

4 Overgroup $\widehat{G}$ and relation with holomorphic dynamics

4.1 Definition of $\widehat{G}$

We will consider here a subgroup of $\widetilde{G}$, which seems to be more tractable. At least it will have a very natural interpretation as an iterated monodromy group of a holomorphic mapping of a 2-dimensional projective complex plane.

Consider the following elements of the group $\widetilde{G}$

$$T = \beta b^{-1} a, \qquad S = \gamma c^{-1} b.$$

Note that $T^{-1} S^{-1} = a^{-1} c \beta^{-1} \gamma^{-1}$.

We have the following relations

$$\begin{aligned} T\alpha T^{-1} &= \beta b^{-1} a \alpha a^{-1} b \beta^{-1} = \beta\alpha\beta^{-1} \\ T\beta T^{-1} &= \beta b^{-1} a \beta a^{-1} b \beta^{-1} = \beta b^{-1} \alpha\beta\alpha^{-1} b \beta^{-1} = \beta\alpha\beta\alpha^{-1}\beta^{-1} \\ T\gamma T^{-1} &= \beta b^{-1} a \gamma a^{-1} b \beta^{-1} = \beta\beta^{-1}\gamma\beta\beta^{-1} = \gamma, \end{aligned}$$

and

$$
\begin{aligned}
S\alpha S^{-1} &= \gamma c^{-1} b\alpha b^{-1} c\gamma^{-1} = \gamma\gamma^{-1}\alpha\gamma\gamma^{-1} = \alpha\\
S\beta S^{-1} &= \gamma c^{-1} b\beta b^{-1} c\gamma^{-1} = \gamma\beta\gamma^{-1}\\
S\gamma S^{-1} &= \gamma c^{-1} b\gamma b^{-1} c\gamma^{-1} = \gamma\beta\gamma\beta^{-1}\gamma^{-1}
\end{aligned}
$$

The elements T and S satisfy the following recursions

$$
T = \pi(a^{-1}c, \alpha a^{-1}c, \alpha, 1) = \pi(T^{-1}S^{-1}\gamma\beta, \alpha T^{-1}S^{-1}\gamma\beta, \alpha, 1) = \pi(T^{-1}S^{-1}\gamma\beta, T^{-1}S^{-1}\gamma\beta\gamma^{-1}\alpha\gamma, \alpha, 1)
$$

If we conjugate the elements α, β, γ by $\delta = \sigma\left(\delta\alpha^{-1}, \delta\gamma, \delta, \delta\gamma\right)$, i.e., replace α, β, γ by $\delta\alpha\delta^{-1}$, $\delta\beta\delta^{-1}$ and $\delta\gamma\delta^{-1}$, then we get the following recursions

$$
\begin{aligned}
\widetilde{\alpha} &= \sigma\left(\widetilde{\alpha}^{-1}, \widetilde{\gamma}\widetilde{\alpha}, 1, \widetilde{\gamma}\right)\\
\widetilde{\beta} &= \left(1, \widetilde{\alpha}, \widetilde{\gamma}\widetilde{\alpha}\widetilde{\gamma}^{-1}, 1\right)\\
\widetilde{\gamma} &= \left(\widetilde{\gamma}\widetilde{\beta}\widetilde{\gamma}^{-1}, 1, \widetilde{\gamma}\widetilde{\beta}\widetilde{\gamma}^{-1}, 1\right)
\end{aligned}
$$

Let us denote $\tau = \widetilde{\gamma}\widetilde{\beta}\widetilde{\alpha}$. We get

$$
\begin{aligned}
\tau = \left(\widetilde{\gamma}\widetilde{\beta}\widetilde{\gamma}^{-1}, 1, \widetilde{\gamma}\widetilde{\beta}\widetilde{\gamma}^{-1}, 1\right)\left(1, \widetilde{\alpha}, \widetilde{\gamma}\widetilde{\alpha}\widetilde{\gamma}^{-1}, 1\right)\sigma\left(\widetilde{\alpha}^{-1}, \widetilde{\gamma}\widetilde{\alpha}, 1, \widetilde{\gamma}\right) =&\\
\sigma\left(\widetilde{\alpha}\widetilde{\alpha}^{-1}, \widetilde{\gamma}\widetilde{\beta}\widetilde{\gamma}^{-1}\widetilde{\gamma}\widetilde{\alpha}, 1, \widetilde{\gamma}\widetilde{\beta}\widetilde{\gamma}^{-1}\widetilde{\gamma}\widetilde{\alpha}\widetilde{\gamma}^{-1}\widetilde{\gamma}\right) = \sigma\left(1, \widetilde{\gamma}\widetilde{\beta}\widetilde{\alpha}, 1, \widetilde{\gamma}\widetilde{\beta}\widetilde{\alpha}\right) = \sigma\left(1, \tau, 1, \tau\right).&
\end{aligned}
$$

The automorphisms S and T will be conjugated to automorphisms satisfying the recursions

$$
\begin{aligned}
S &= (1, 1, T, T)\\
T &= \pi\left(T^{-1}S^{-1}\tau, T^{-1}S^{-1}\tau, 1, 1\right),
\end{aligned}
$$

where $\pi = (13)(24)$.

In the next proposition the letters $1, 2, 3, 4$ are, as before, identified with the pairs $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, respectively, so that the subgroup G of $\widehat{G}$ acts only on the first coordinates of the letters.

Proposition 4.1. *The group $\widehat{G}$ is level-transitive. The subgroup G is normal and coincides with the set of elements $g \in \widehat{G}$ such that*

$$
g\left((x_1, y_1)(x_2, y_2)\ldots(x_n, y_n)\right) = (x_1', y_1)(x_2', y_2)\ldots(x_n', y_n),
$$

i.e., elements acting trivially on the second coordinates of the letters.

The quotient $\widehat{G}/G$ is isomorphic to the group acting on the binary tree generated by the transformations

$$
s = (1, t), \quad t = \sigma\left(t^{-1}s^{-1}, 1\right),
$$

Moreover, $\widehat{G}/G$ is isomorphic to the subgroup $\langle S, T\rangle$ of $\widehat{G}$ and $\widehat{G} = G \rtimes \langle S, T\rangle$ and to the subgroup $\left\langle g_1 g_0^{-1}, g_2 g_1^{-1}\right\rangle$ of $\widetilde{H}$.

Proof. We will use here the recurrence

$$
\begin{aligned}
\alpha &= \sigma\left(\alpha^{-1}, \gamma\alpha, 1, \gamma\right)\\
\beta &= \left(1, \alpha, \gamma\alpha\gamma^{-1}, 1\right)\\
\gamma &= \left(\gamma\beta\gamma^{-1}, 1, \gamma\beta\gamma^{-1}, 1\right)\\
S &= (1, 1, T, T)\\
T &= \pi\left(T^{-1}S^{-1}\tau, T^{-1}S^{-1}\tau, 1, 1\right),
\end{aligned}
$$

where $\tau = \gamma\beta\alpha$.

The group $\widehat{G}$ is level-transitive, since it is transitive on the first level and the wreath recursion is recurrent.

Let $N \le \widehat{G}$ be the subgroup of the elements acting trivially on the second coordinates. Then N is normal and it follows from the recursion, defining S and T that $\widehat{G}/N$ is isomorphic to the group $H \le \widetilde{H}$ generated by the automorphisms

$$s = g_1 g_0^{-1} = (1, t)\,, \quad t = g_2 g_1^{-1} = \sigma\left(t^{-1}s^{-1}, 1\right) \tag{1}$$

of the binary tree. The canonical homomorphism $\widehat{G} \to H$ is induced by the map $(\mathsf{X} \times \mathsf{X})^* \to \mathsf{X}^*$ ignoring the first coordinates of the letters.

Lemma 4.2. *The abelianization of H is $\mathbb{Z}^2$, where the images of s and t are free generators of $\mathbb{Z}^2$.*

Proof. The proof is the same as in Lemma 3.8. The corresponding linear map T is in our case given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. The group H is contracting, since it is a subgroup of $\widetilde{H}$. □

The subgroup $G \le \widehat{G}$ is a normal, since it is normal in $\widetilde{G}$. It follows from the description of the conjugation of G by S and T that every element of $\widehat{G}$ can be written in the form $h \cdot g$, where $h \in \langle S, T\rangle$ and $g \in G = \langle \alpha, \beta, \gamma\rangle$. We are going to prove that H coincides with $\langle S, T\rangle$ so that $\widehat{G} = G \rtimes \langle S, T\rangle$. This will show that $N = G$.

Suppose that the image of $h \in \langle S, T\rangle$ in H is trivial. We have to prove that $h = 1$. The wreath recursion (1) is contracting, hence the kernel of the canonical epimorphism from the free group onto H coincides with the union of the kernels $\mathcal{E}_n$ of the iterations of the wreath recursions.

The generators of the first level stabilizer in $\langle S, T\rangle$ are

$$\begin{aligned} S &= (1, 1, T, T) \\ T^2 &= \left(T^{-1}S^{-1}\tau, T^{-1}S^{-1}\tau, T^{-1}S^{-1}\tau, T^{-1}S^{-1}\tau\right) \\ TST^{-1} &= (T, T, 1, 1). \end{aligned}$$

It is easy to check that τ commutes with T and S. We conclude that if $h = (h_1, h_2, h_3, h_4)$ belongs to the first level stabilizer, then $h_1, h_2, h_3, h_4 \in \langle S, T\rangle \cdot \tau^k$, where k is half of the sum of powers of T in h, i.e., is half of one of the coordinates of the image of h in the abelianization $\mathbb{Z}^2$. Consequently, by the above lemma, if $h = (h_1, h_2, h_3, h_4) \in \mathcal{E}_n$ is trivial in H, then h_1, h_2, h_3, h_4 belong to $\langle S, T\rangle$ and $h_i \in \mathcal{E}_{n-1}$. We get, by induction, that if $h \in \langle S, T\rangle$ is trivial in H, then it is trivial in $\widehat{G}$. □

4.2 G_w as iterated monodromy groups

Let $C_0, C_1, \ldots,$ be a sequence of planes and let $\mathrm{A}_i, \mathrm{B}_i, \Gamma_i \in C_i$ be three pairwise different points in the respective plane. Let $f_i : C_i \to C_{i-1}$, for $i = 1, 2, \ldots$, be an orientation-preserving 2-fold branched covering with critical point Γ_i such that

$$\begin{aligned} f_i(\Gamma_i) &= \mathrm{A}_{i-1} \\ f_i(\mathrm{A}_i) &= \mathrm{B}_{i-1} \\ f_i(\mathrm{B}_i) &= \Gamma_{i-1}. \end{aligned}$$

We denote $\mathcal{M}_i = C_i \setminus \{\mathrm{A}_i, \mathrm{B}_i, \Gamma_i\}$. See Figure 4.

Let $t \in \mathcal{M}_0$. Denote by L_n the set of preimages of the point t under the composition $f_1 \circ f_2 \circ \cdots \circ f_n : C_n \to C_0$. We obviously have $|L_n| = 2^n$. The union $T = \bigsqcup_{n=0}^{\infty} L_n$ has a natural structure of a rooted binary tree, where a vertex $t_n \in L_n$ is connected by an edge with the vertex $f_n(t_n) \in L_{n-1}$. The root is $t \in L_0$ and the set L_n is the nth level of the tree T.

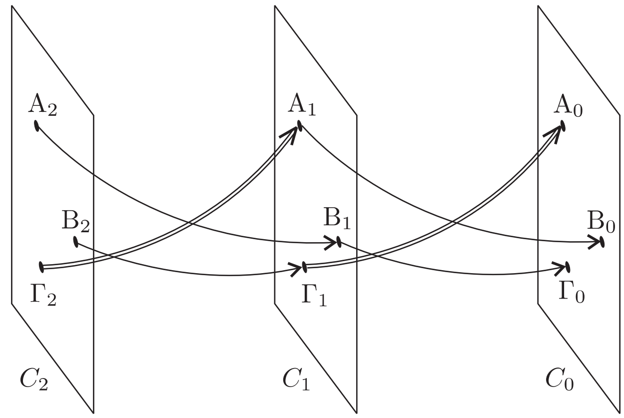

Figure 4: Backward iteration

The fundamental group $\pi_1(\mathcal{M}_0, t)$ acts naturally on each of the levels of the tree T and this action agrees with the tree structure. The obtained automorphism group of the tree T is called *iterated monodromy group* of the sequence $f_1, f_2, \ldots$, and is denoted $\mathrm{IMG}\,(f_1, f_2, \ldots)$.

Let α be a small simple loop going in positive direction around the point A_0 connected to the basepoint t by a path. Similarly, we define the elements β and γ of $\pi_1(\mathcal{M}_0, t)$ as small simple positive loops around B_0 and Γ_0, respectively.

It is easy to see, just looking at the tree of preimages of the points A_0, B_0 and Γ_0 and at the branching degrees, that the respective elements α, β and γ of the iterated monodromy group are conjugate to the elements g_0, g_1, g_2, respectively.

Consequently, $\mathrm{IMG}\,(f_1, f_2, \ldots)$ is isomorphic to G_w for some $w \in \{0,1\}^\omega$.

Let us describe, how one can find an appropriate sequence $w = x_1 x_2 \ldots$ (which is obviously not unique).

Let $l_{\mathrm{A}_0}, l_{\mathrm{B}_0}$ and l_{Γ_0} be simple disjoint paths in $\mathcal{M}_0$ connecting infinity with the points A_0, B_0 and Γ_0, respectively.

The f_1-preimage of the path l_{A_0} is a path, passing through Γ_1 and dividing the plane $\mathbb{C}_1$ into two pieces. The point A_1 belongs to one of these pieces, which we will denote S_1. The other piece will be denoted S_0.

We have then two possibilities: either B_1 belongs to S_1, or it belongs to S_0. In the first case we put $x_1 = 0$, in the second $x_1 = 1$.

The preimage $f_1^{-1}\left(l_{\mathrm{B}_0}\right)$ is a disjoint union of two paths, connecting the preimages of B_0 to infinity. One of these paths is a path connecting A_1 to infinity. We will denote it l_{A_1}. Similarly, l_{B_1} is the preimage of l_{Γ_0}, connecting B_1 to infinity.

The point Γ_1 divides the path $f_1^{-1}\left(l_{\mathrm{A}_0}\right)$ into two halves. If you go around Γ_1 in positive direction, then you cross one of the halves, when coming from S_1 to S_0 and the other half, when coming from S_0 to S_1. Let l_{Γ_1} be the first half.

We get in this way a collection $\{l_{\mathrm{A}_1}, l_{\mathrm{B}_1}, l_{\Gamma_1}\}$ (a *spider*) of the paths connecting infinity with the points A_1, B_1 and Γ_1 in the plane C_1. We can use these paths, in the same way as we used the paths $\{l_{\mathrm{A}_0}, l_{\mathrm{B}_0}, l_{\Gamma_0}\}$, to construct the next spider $\{l_{\mathrm{A}_2}, l_{\mathrm{B}_2}, l_{\Gamma_2}\}$, to partition the plane C_2 into two parts S_0, S_1, and to get the next letter x_2 of the word w.

We proceed inductively, and get spiders $\{l_{\mathrm{A}_i}, l_{\mathrm{B}_i}, l_{\Gamma_i}\}$ such that

1. $l_{\mathrm{A}_i} \subset f_i^{-1}\left(l_{\mathrm{B}_{i-1}}\right)$, $l_{\mathrm{B}_i} \subset f_i^{-1}\left(l_{\Gamma_{i-1}}\right)$, $l_{\Gamma_i} \subset f_i^{-1}\left(l_{\mathrm{A}_{i-1}}\right)$,

2. the curve $f_i^{-1}\left(l_{\mathrm{A}_{i-1}}\right)$ divides the plane C_i into two parts S_0, S_1, where the point A_i belongs to S_1,

3. if $\mathrm{B}_i \in S_1$, then $x_i = 0$, otherwise $x_i = 1$,

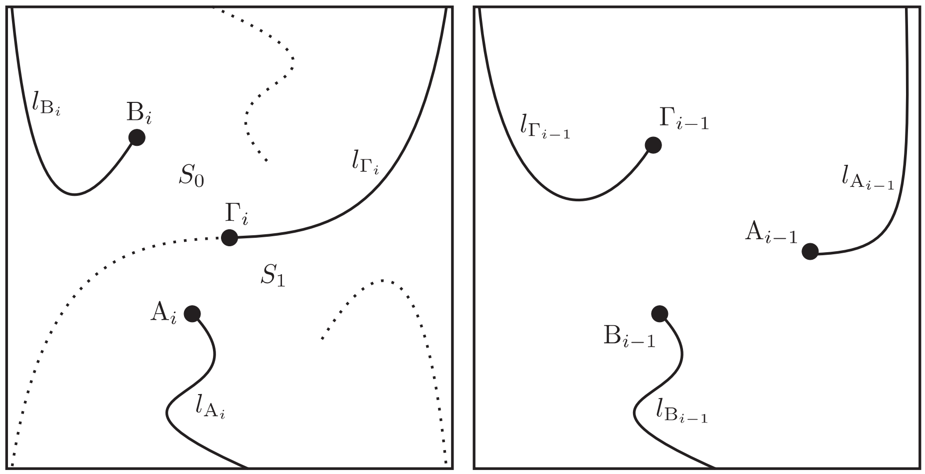

Figure 5: Spiders

4. if one goes in the positive direction around Γ_i, then the path l_{Γ_i} is crossed when coming from S_1 to S_0.

See Figure 5 for a picture of these curves.

Proposition 4.3. *Let $\{l_{A_i}, l_{B_i}, l_{\mathbb{C}_i}\}$ be the spiders, constructed above and let $w = x_1x_2\dots$ be the obtained sequence. Then the iterated monodromy group* $\mathrm{IMG}(f_1, f_2, \dots)$ *is isomorphic to G_w.*

Proof. For every $t_n \in L_n$, let $\Lambda(t_n) = a_1a_2\dots a_n$ be the *itinerary* of t_n, i.e., such a sequence that $a_i = 0$ if $f_i \circ \dots \circ f_n(t_n)$ belongs to the sector S_0 of the plane C_i and $a_i = 1$ if it belongs to S_1. We set $\Lambda(t) = \emptyset$.

It follows directly from the definition that if $\Lambda(t_n) = a_1a_2\dots a_n$, then $\Lambda(f_n(t_n)) = a_1a_2\dots a_{n-1}$. This easily implies, that the map $\Lambda : T \to \{0,1\}^*$ is an isomorphism of the rooted trees.

If $x, y \in C_i \backslash (l_{A_i} \cup l_{B_i} \cup l_{\Gamma_i})$, and $\Delta \in \{A_i, B_i, \Gamma_i\}$, then we denote by $\ell_\Delta(x, y)$ the path starting in x, ending in y and crossing the spider $l_{A_i} \cup l_{B_i} \cup l_{\Gamma_i}$ only once, intersecting it along l_Δ in positive direction (i.e., so that the point Δ is on the left-hand side from l_Δ at the intersection point with $\ell_\Delta(x, y)$). The curve $\ell_\Delta(x, y)$ is uniquely defined, up to a homotopy in $\mathcal{M}_i$. We also denote by $\ell_i(x, y)$ the path from x to y disjoint with the spider. It is also well defined, up to a homotopy.

For a given point $x \in C_i \setminus (l_{A_i} \cup l_{B_i} \cup l_{\Gamma_i})$ we denote by x_0 the f_{i+1}-preimage of x, belonging to S_0 and by x_1 the preimage, belonging to S_1.

It follows then from the definitions that

$$\begin{aligned}
f_i^{-1}\left(\ell_{A_{i-1}}(x, y)\right) &= \ell_{\Gamma_i}(x_1, y_0) \cup \ell_i(x_0, y_1)\\
f_i^{-1}\left(\ell_{B_{i-1}}(x, y)\right) &= \ell_{A_i}(x_1, y_1) \cup \ell_i(x_0, y_0)\\
f_i^{-1}\left(\ell_{\Gamma_{i-1}}(x, y)\right) &= \begin{cases} \ell_{B_i}(x_1, y_1) \cup \ell_i(x_0, y_0) & \text{if } x_i = 0\\ \ell_{B_i}(x_0, y_0) \cup \ell_i(x_1, y_1) & \text{if } x_i = 1\end{cases}\\
f_i^{-1}\left(\ell_{i-1}(x, y)\right) &= \ell_i(x_0, y_0) \cup \ell_i(x_1, y_1).
\end{aligned}$$

Let $\alpha = \ell_{A_0}(t, t)$, $\beta = \ell_{B_0}(t, t)$ and $\gamma = \ell_{\Gamma_0}(t, t)$. An easy inductive argument shows that the isomorphism $\Lambda : T \to \{0,1\}^*$ conjugates the action of α, β, γ on T with α_w, β_w and γ_w, respectively. □

It is also easy to deduce from the proof of the proposition that every group G_w can be realized as the iterated monodromy group of a sequence $f_1, f_2, \dots$. We will also prove this later using other methods.

4.3 The group $\widehat{G}$ as an iterated monodromy group

Let us choose any complex structure on the plane C_0, identifying it with $\mathbb{C}$. The complex structures on C_i is defined then as the pull-back of the complex structure on C_0 by the map $f_1 \circ f_2 \circ \cdots f_i$. Then the maps f_i become quadratic polynomials. Applying the respective affine transformations, we may assume that Γ_i coincides with 0 and A_i coincides with $1 \in \mathbb{C}$. Let us denote by p_i the complex number, identified then with B_i. Then f_i is a quadratic polynomial such that

1. its critical point is 0,
2. $f_i(0) = 1$,
3. $f_i(1) = p_{i-1}$,
4. and $f_i(p_i) = 0$.

The first two conditions imply that f_i is a quadratic polynomial of the form $az^2 + 1$. The last condition implies that $a = -\frac{1}{p_i^2}$.

We conclude that $p_{i-1} = 1 - \frac{1}{p_i^2}$ and $f_i(z) = 1 - \frac{z^2}{p_i^2}$.

We get thus a map

$$\left(\begin{array}{c} z \\ p \end{array}\right) \mapsto \left(\begin{array}{c} 1 - \frac{z^2}{p^2} \\ 1 - \frac{1}{p^2} \end{array}\right).$$

It can be considered as a map from $\mathbb{CP}^2$ to itself, defined in the homogeneous coordinates by the formula

$$F : [z : p : u] \mapsto \left[p^2 - z^2 : p^2 - u^2 : p^2\right].$$

The Jacobian of the map F is

$$\left|\begin{array}{ccc} -2z & 0 & 0 \\ 2p & 2p & 2p \\ 0 & -2u & 0 \end{array}\right| = -8zpu,$$

hence the critical set is the union of the three lines $z = 0, p = 0$ and $u = 0$.

The orbits of these lines are

$$\{z = 0\} \mapsto \{z = u\} \mapsto \{z = p\} \mapsto \{z = 0\},$$

$$\{p = 0\} \mapsto \{u = 0\} \mapsto \{p = u\} \mapsto \{p = 0\}.$$

Hence, the post-critical set of the map F is the union of the lines $p = 0, z = 0, u = 0, p = z, p = u, z = u$. (Or, in non-homogeneous coordinates, the lines $p = 0, z = 0, p = 1, z = 1, p = z$ and the line at infinity.) Thus, it is post-critically finite. It also follows that F is hyperbolic, i.e., expanding on a neighborhood of its Julia set, since the complement $\mathcal{M}$ of these lines in $\mathbb{CP}^2$ is Kobayashi hyperbolic.

Theorem 4.4. *The group* $\mathrm{IMG}\,(F)$ *is isomorphic to the group* $\widehat{G}$. *The group* $\mathrm{IMG}\left(1 - \frac{1}{p^2}\right)$ *is isomorphic to the quotient* $\widehat{G}/G = H$.

Proof. Let $\mathcal{M}$ be the complement of the post-critical set of F in $\mathbb{CP}^2$. The space $\mathcal{M}$ can be interpreted as the configuration space of pairs (z, p) of points of $\mathbb{C}$ different from 0 and 1 and different from each other.

Every given value of p defines the function $f_p(z) = 1 - \frac{z^2}{p^2}$ as a polynomial over z. Its critical point is 0 and

$$f_p(0) = 1$$

$$f_p(1) = 1 - \frac{1}{p^2}$$

$$f_p(p) = 0.$$

Thus the polynomial f_p maps the points $(0, 1, z, p)$ to the points $\left(1, 1 - \frac{1}{p^2}, 1 - \frac{z^2}{p^2}, 0\right)$. The second and the third value are precisely the second and the first coordinate of $F(z, p)$, respectively.

One can use this interpretation of the map $F : F^{-1}(\mathcal{M}) \to \mathcal{M}$ to compute IMG (F).

The map $p \mapsto 1 - \frac{1}{p^2}$ has three fixed points. They are the roots of the polynomial $p^3 - p^2 + 1$ and are equal approximately to $0.8774 \pm 0.7449i$ and -0.7549. The corresponding polynomial f_p will be post-critically finite with the critical dynamics of the form

$$0 \mapsto 1 \mapsto p \mapsto 0.$$

The polynomial f_p for $p \approx 0.8774+0.7449i$ is called "Douady rabbit", for $p \approx -0.7549$ is called "airplane". These names originate from the shape of their Julia sets (see Figure 6). Let us call f_p for $p \approx 0.8774 - 0.7449i$ "co-rabbit".

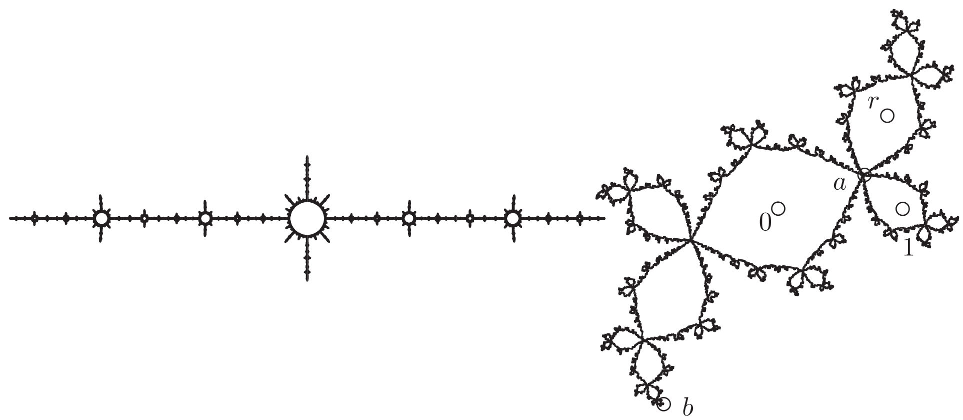

Figure 6: Airplane and Rabbit

Let $r \approx 0.8774+0.7449i$ be the value of the parameter p, corresponding to the rabbit polynomial $1 - \frac{z^2}{r}$. This polynomial has two fixed points: $a \approx 0.6865 + 0.2580i$ and $b \approx -0.9016 - 1.5652i$. See their position relatively to the Julia set of the rabbit on right-hand side of Figure 6.

Let us take the pair $(z, p) = (a, r)$ as a basepoint of the space $\mathcal{M}$. It is a fixed point under F and its F-preimages are the four pairs $(a, r), (-a, r), (a, -r)$ and $(-a, -r)$. The fundamental group of the space $\mathcal{M}$ is generated by the elements $\alpha, \beta, \gamma, S, T$. Here α, β, γ are the loops in the configuration space $\mathcal{M}$ obtained by moving the point z around the points $1, r$ and 0, respectively, as it is shown on Figure 7, the point p being fixed at r. The generators S and T are loops obtained by moving the point p around 0 and 1, respectively, as it is also shown on Figure 7, the point z being fixed at a.

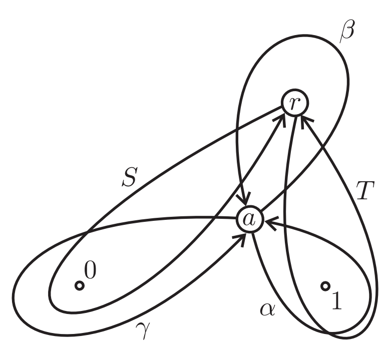

Figure 7: Generators of IMG (F)

It is known that the fundamental group of $\mathcal{M}$ is (a quotient of?) the semidirect product of the free groups $\langle \alpha, \beta, \gamma \rangle \rtimes \langle S, T \rangle$, where the action is given by

$$\begin{aligned} T\alpha T^{-1} &= \beta\alpha\beta^{-1}, & S\alpha S^{-1} &= \alpha, \\ T\beta T^{-1} &= \beta\alpha\beta\alpha^{-1}\beta^{-1}, & S\beta S^{-1} &= \gamma\beta\gamma^{-1}, \\ T\gamma T^{-1} &= \gamma, & S\gamma S^{-1} &= \gamma\beta\gamma\beta^{-1}\gamma^{-1}, \end{aligned}$$

since S and T correspond to the Dehn twists around the simple curves, shown on Figure 8. It is also known that the subgroup $\langle \alpha, \beta, \gamma \rangle$ has trivial centralizer in $\langle \alpha, \beta, \gamma, S, T \rangle$, hence every element g of $\pi_1(\mathcal{M})$ is uniquely defined by the elements $g\alpha g^{-1}, g\beta g^{-1}$ and $g\gamma g^{-1}$.

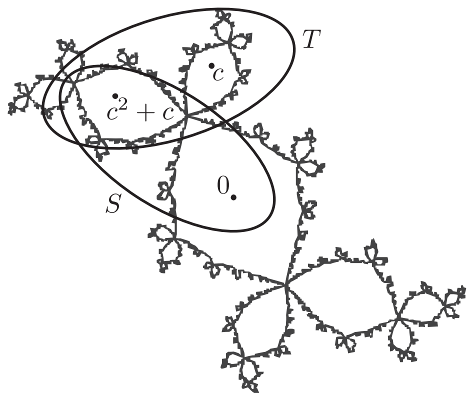

Figure 8: Dehn twists S and T

We will compute the associated virtual endomorphism ϕ. The domain of the virtual endomorphism is the subgroups of the loops in $\mathcal{M}$ such that their F-preimage starting at the basepoint (a, r) is again a loop. It is a subgroup of index 4 in $\pi_1(\mathcal{M}, (a, r))$ and is generated by $\alpha^2, \beta, \gamma, \alpha\beta\alpha^{-1}, \alpha\gamma\alpha^{-1}, S, T^2$ and TST^{-1}.

Figure 9 shows the f_r-preimages of the loops α, β, γ and the Dehn twists S, T. It follows that

$$\begin{aligned} \phi\left(\alpha^2\right) &= \gamma, & \phi\left(\alpha\beta\alpha^{-1}\right) &= 1 \\ \phi(\beta) &= \alpha, & \phi\left(\alpha\gamma\alpha^{-1}\right) &= 1 \\ \phi(\gamma) &= \beta, & \phi(S) &= 1 \end{aligned}$$

and that $R = \phi\left(T^2\right)$ is equal to the Dehn twist along the curve going around the points 0 and 1.

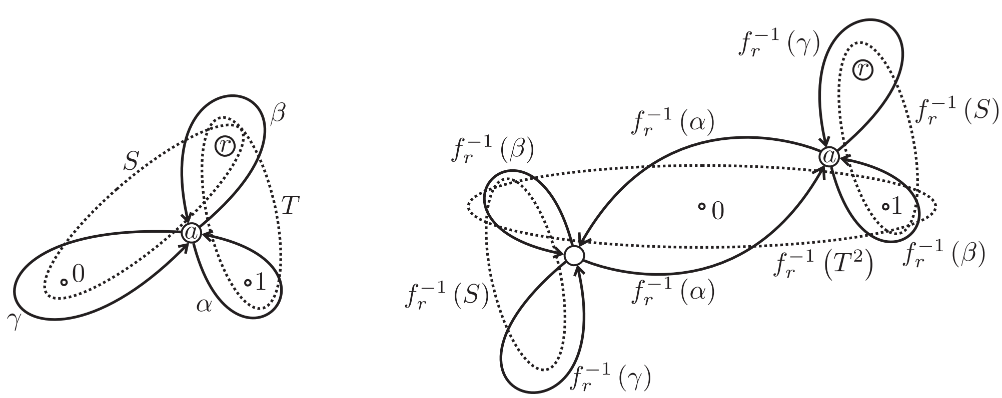

Figure 9: Generators and their preimages

We have

$$\begin{aligned}
R\alpha R^{-1} &= \alpha\gamma\alpha\gamma^{-1}\alpha^{-1} \\
R\beta R^{-1} &= \beta \\
R\gamma R^{-1} &= \alpha\gamma\alpha^{-1},
\end{aligned}$$

which implies that $R = T^{-1}S^{-1}\gamma\beta\gamma^{-1}\alpha\gamma$. This can be proved showing that $T^{-1}S^{-1}\gamma\beta\gamma^{-1}\alpha\gamma$ acts in the same way by conjugation on $\langle\alpha,\beta,\gamma\rangle$, as R does. Similar topological considerations show that $\phi\left(TST^{-1}\right) = 1$.

We can, however, find $\phi(S), \phi\left(T^2\right)$ and $\phi\left(TST^{-1}\right)$ algebraically in the following way.

We have

$$\begin{gathered}
\phi\left(S\right)\alpha\phi\left(S\right)^{-1} = \phi\left(S\beta S^{-1}\right) = \phi\left(\gamma\beta\gamma^{-1}\right) = \beta\alpha\beta^{-1}, \\
\phi\left(S\right)\beta\phi\left(S\right)^{-1} = \phi\left(S\gamma S^{-1}\right) = \phi\left(\gamma\beta\gamma\beta^{-1}\gamma^{-1}\right) = \beta\alpha\beta\alpha^{-1}\beta^{-1}, \\
\phi\left(S\right)\gamma\phi(S)^{-1} = \phi\left(S\alpha^2 S^{-1}\right) = \phi\left(\alpha^2\right) = \gamma,
\end{gathered}$$

which implies that $\phi(S) = T$.

We have

$$\begin{gathered}
\phi\left(T^2\right)\alpha\phi\left(T^2\right)^{-1} = \phi\left(T^2\beta T^{-1}\right) = \phi\left(\beta\alpha\beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\beta^{-1}\right) = \alpha\gamma\alpha\gamma^{-1}\alpha^{-1}, \\
\phi\left(T^2\right)\beta\phi\left(T^2\right)^{-1} = \phi\left(T^2\gamma T^{-2}\right) = \phi\left(\gamma\right) = \beta, \\
\phi\left(T^2\right)\gamma\phi\left(T^2\right)^{-1} = \phi\left(T^2\alpha^2 T^{-2}\right) = \phi\left(\beta\alpha\beta\alpha^2\beta^{-1}\alpha^{-1}\beta^{-1}\right) = \alpha\gamma\alpha^{-1},
\end{gathered}$$

which implies that $\phi\left(T^2\right) = R = T^{-1}S^{-1}\gamma\beta\gamma^{-1}\alpha\gamma$.

Finally,

$$\begin{gathered}
\phi\left(TST^{-1}\right)\alpha\phi\left(TST^{-1}\right)^{-1} = \phi\left(TST^{-1}\beta TS^{-1}T^{-1}\right) = \phi\left(\beta\right) = \alpha, \\
\phi\left(TST^{-1}\right)\beta\phi\left(TST^{-1}\right)^{-1} = \phi\left(TST^{-1}\gamma TS^{-1}T^{-1}\right) = \phi\left(\gamma\beta\alpha\beta\alpha^{-1}\beta^{-1}\gamma\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}\right) = \beta,
\end{gathered}$$

and

$$\begin{aligned}
&\phi\left(TST^{-1}\right)\gamma\phi\left(TST^{-1}\right)^{-1} = \phi\left(TST^{-1}\alpha^2 TS^{-1}T^{-1}\right) = \\
&\qquad \phi\left(\beta\alpha^{-1}\beta^{-1}\gamma\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}\beta\alpha^2\beta^{-1}\gamma\beta\alpha\beta\alpha^{-1}\beta^{-1}\gamma^{-1}\beta\alpha\beta^{-1}\right) = \gamma,
\end{aligned}$$

hence $\phi\left(TST^{-1}\right) = 1$.

We know that the group $\widehat{G}$ is given by the recursion

$$\begin{aligned}
\alpha &= \sigma\left(1,\gamma,1,\gamma\right) \\
\beta &= \left(\alpha,1,1,\alpha\right) \\
\gamma &= \left(1,\beta,1,\beta\right), \\
S &= \left(1,1,T,T\right) \\
T &= \pi\left(T^{-1}S^{-1}\gamma\beta, T^{-1}S^{-1}\gamma\beta\gamma^{-1}\alpha\gamma, \alpha, 1\right).
\end{aligned}$$

We have then

$$\begin{aligned}
\alpha^2 &= \left(\gamma,\gamma,\gamma,\gamma\right) \\
\alpha\beta\alpha^{-1} &= \left(1,\alpha,\gamma\alpha\gamma^{-1},1\right) \\
\alpha\gamma\alpha^{-1} &= \left(\gamma\beta\gamma^{-1},1,\gamma\beta\gamma^{-1},1\right), \\
S &= \left(1,1,T,T\right) \\
T^2 &= \left(\alpha T^{-1}S^{-1}\gamma\beta, T^{-1}S^{-1}\gamma\beta\gamma^{-1}\alpha\gamma, T^{-1}S^{-1}\gamma\beta\alpha, T^{-1}S^{-1}\gamma\beta\gamma^{-1}\alpha\gamma\right), \\
TST^{-1} &= \left(\alpha T\alpha^{-1}, T, 1, 1\right).
\end{aligned}$$

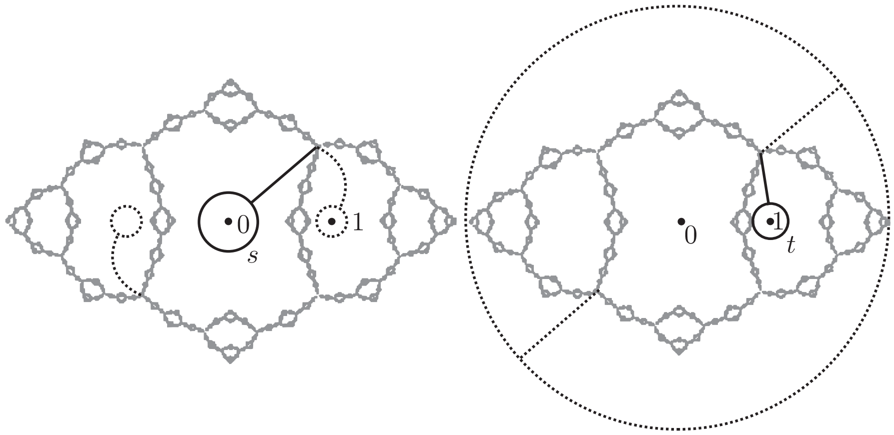

Figure 10: Computation of $\mathrm{IMG}\left(1-\frac{1}{p^2}\right)$

We see that the virtual endomorphism of $\pi_1(\mathcal{M})$, associated with the self-covering F coincides with the virtual endomorphism of the self-similar group $\widehat{G}$, associated to the last coordinate of the wreath recursion. This proves that $\mathrm{IMG}(F)$ coincides with $\widehat{G}$, since the associated virtual endomorphism determines the iterated monodromy group uniquely.

Let us show that $\widehat{G}/G \cong \mathrm{IMG}\left(1-\frac{1}{p^2}\right)$ directly, computing the standard action of $\mathrm{IMG}\left(1-\frac{1}{p^2}\right)$ on X^*. Another proof is just to use the semi-conjugacy of F with $\mathrm{IMG}\left(1-\frac{1}{p^2}\right)$ (obtained by projecting F onto the second coordinate) and Proposition 4.1.

Let us take the fixed point $r \approx 0.8774+0.7449i$ of $1-\frac{1}{p^2}$ as the basepoint. It has two preimages: itself and $-r$. Let ℓ_1 be the trivial path at r and let ℓ_0 be the path connecting r with $-r$ and going above the puncture 0. Then we get the following wreath recursions for the corresponding standard action (see Figure 10 for the loops S, T, their preimages and the Julia set):

$$S = (1,\ T), \qquad T = \sigma(T^{-1}S^{-1},\ 1),$$

which coincides with the wreath recursion for $H = \widehat{G}/G$. $\square$

Note that the group $H = \mathrm{IMG}\left(1-\frac{1}{z^2}\right)$ is a subgroup of the iterated monodromy group $G_{111\ldots}$ of the rabbit. This implies existence of a surjective continuous map from the Julia set of $1-\frac{1}{z^2}$ to the "Douady rabbit", which is a semi-conjugacy of the respective topological dynamical systems.

4.4 Limit spaces of G and $\widehat{G}$

Since the complement of the post-critical set of $F : (z,p) \mapsto \left(1-\frac{z^2}{p^2}, 1-\frac{1}{p^2}\right)$ is Kobayashi hyperbolic, the map F is expanding with respect to some Riemann metric on a neighborhood of its Julia set, and therefore its iterated monodromy group $\mathrm{IMG}(F) = \widehat{G}$ is contracting. The group $G < \widehat{G}$ is hence also contracting.

The limit space $\mathcal{J}_{\widehat{G}}$ is homeomorphic, by [5, Theorem 5.5.3], to the Julia set of F.

Recall, that the limit space $\mathcal{J}_{\widehat{G}}$ is the quotient of the space of left-infinite sequences $(\mathsf{X}\times\mathsf{X})^{-\omega}$ by the *asymptotic equivalence relation.* Two sequences $\ldots(x_2,y_2)(x_1,y_1)$ and $\ldots(x_2',y_2')(x_1',y_1')$ are asymptotically equivalent if and only if there exists a sequence $g_n \in \widehat{G}$, assuming only a finite number of different values, such that

$$g_n\left((x_n,y_n)\ldots(x_2,y_2)(x_1,y_1)\right) = (x_n',y_n')\ldots(x_2',y_2')(x_1',y_1').$$

Proposition 4.5. *We have the following commutative diagram of continuous surjective maps*

$$\begin{array}{ccc} \mathcal{J}_G & \to & \mathcal{J}_{\widehat{G}} \\ \downarrow & & \downarrow \\ \mathsf{X}^{-\omega} & \to & \mathcal{J}_H \end{array}$$

where the maps $\mathcal{J}_G \to \mathcal{J}_{\widehat{G}}$ and $\mathsf{X}^{-\omega} \to \mathcal{J}_H$ are induced by the inclusions of self-similar groups $G < \widehat{G}$ and $\{1\} < H$, respectively, and the vertical arrows are induced by the map $(\mathsf{X} \times \mathsf{X})^{-\omega} \to \mathsf{X}^{-\omega}$, projecting the letters to their second coordinates (and the respective epimorphisms of groups).

Moreover, if J is the preimage of a point $\zeta \in \mathcal{J}_H$ under the surjection $\mathcal{J}_{\widehat{G}} \to \mathcal{J}_H$, then preimage of J under the surjection $P : \mathcal{J}_G \to \mathcal{J}_{\widehat{G}}$ is a disjoint union of a finite number of sets $J_1, J_2, \ldots, J_k$, which are fibers of the projection $\mathcal{J}_G \to \mathsf{X}^{-\omega}$ and are such that $P : J_i \to J$ is a homeomorphism.

Connected components of $\mathcal{J}_G$ are precisely the fibers of the projection $\mathcal{J}_G \to \mathsf{X}^{-\omega}$.

Proof. Commutativity of the diagram follows directly from the commutativity of the corresponding diagram of homomorphisms of self-similar groups and Proposition 4.1.

Let us prove the statement about the fibers of the vertical arrows. It follows from the commutativity of the diagram that if $w_1, \ldots, w_k \in \mathsf{X}^{-\omega}$ are the preimages of a point $\xi \in \mathcal{J}_H$, and $J_1, \ldots, J_k$ are the preimages of the points $w_1, \ldots, w_k$ under the map $\mathcal{J}_G \to \mathsf{X}^{-\omega}$, then $J_1 \sqcup \cdots \sqcup J_k$ is equal to the preimage of J in $\mathcal{J}_G$. If $w_i = \ldots y_2 y_1$, then the preimage J_i of w_i consists of the points of $\mathcal{J}_G$, which are represented by the sequences of the form $\ldots (x_2, y_2)(x_1, y_1) \in (\mathsf{X} \times \mathsf{X})^{-\omega}$.

We have to prove that for every pair $1 \le i, j \le k$ and every point $\xi_i \in J_i$ there exists a unique point $\xi_j \in J_j$ such that the images of ξ_i and ξ_j in $\mathcal{J}_{\widehat{G}}$ are equal. This will prove that the restriction of P onto every J_i is a bijection with J, and thus a homeomorphism.

The sequences $w_i = \ldots y_2' y_1'$ and $w_j = \ldots y_2'' y_1'' \in \mathsf{X}^{-\omega}$ are asymptotically equivalent with respect to H. Consequently, there exists a bounded sequence $h_n \in H = \widehat{G}/G$ such that

$$h_n \left(y_n' \ldots y_1'\right) = y_n'' \ldots y_1''.$$

Let $g_n \in \widehat{G}$ be a bounded sequence such that h_n is the image of g_n under the canonical homomorphism $\widehat{G} \to H$. Let ξ_i be represented by a sequence $\ldots (x_2', y_2')(x_1', y_1')$. Then every partial limit in $\mathsf{X}^* \sqcup \mathsf{X}^{-\omega}$ of the sequence of words $g_n\left((x_n', y_n') \ldots (x_1', y_1')\right)$ is an infinite word of the form $\ldots (x_2'', y_2'')(x_1'', y_1'')$ and is asymptotically equivalent to $\ldots (x_2', y_1')(x_1', y_1')$ with respect to $\widehat{G}$. Hence, every point $\xi_i \in J_i$ is glued with some point $\xi_j \in J_j$ by the map $P : \mathcal{J}_G \to \mathcal{J}_{\widehat{G}}$.

Suppose that $\xi_i \in J_i$ and $\xi_j', \xi_j'' \in J_j$ such that $P\left(\xi_j'\right) = P\left(\xi_j''\right) = P\left(\xi_i\right)$. Let ξ_j' and ξ_j'' be represented by the sequences $\ldots (a_2, y_2'')(a_1, y_1'')$ and $\ldots (b_2, y_2'')(b_1, y_1'')$, respectively. Then these two sequences are asymptotically equivalent with respect to the action of the group $\widehat{G}$, i.e., there exists a sequence g_n of the elements of the nucleus of $\widehat{G}$ such that

$$g_n \cdot (a_n, y_n'') = (b_n, y_n'') \cdot g_{n-1}$$

for every n.

Let h_n be the images of g_n in $H = \widehat{G}/G$. Then we have

$$h_n \cdot y_n'' = y_n'' \cdot h_{n-1}.$$

But the orbifold $\mathcal{J}_H$ has no singular points. This can be shown either directly, or just mentioning that the rational function $1 - \frac{1}{p^2}$ is hyperbolic, thus its Julia set contains no post-critical points. This implies that $h_n = 1$ for all n, since, otherwise, h_n would be a non-trivial element of the isotropy group of the point represented by $\ldots y_{n+2}'' y_{n+1}''$. Hence $g_n \in G$ for all n, which shows that the points ξ_j' and ξ_j'' of $J_j \subset \mathcal{J}_G$ coincide.

It remains to prove the statement about the connected components of $\mathcal{J}_G$. It is sufficient to prove that the fibers of the map $\mathcal{J}_G \to \mathsf{X}^{-\omega}$ are connected. This follows from the level transitivity of the group G on the second coordinates of the words of $(\mathsf{X} \times \mathsf{X})^*$, i.e., from the level transitivity of the groups G_w, in the same way, as connectivity of the limit spaces follows from the level transitivity of the self-similar groups. □

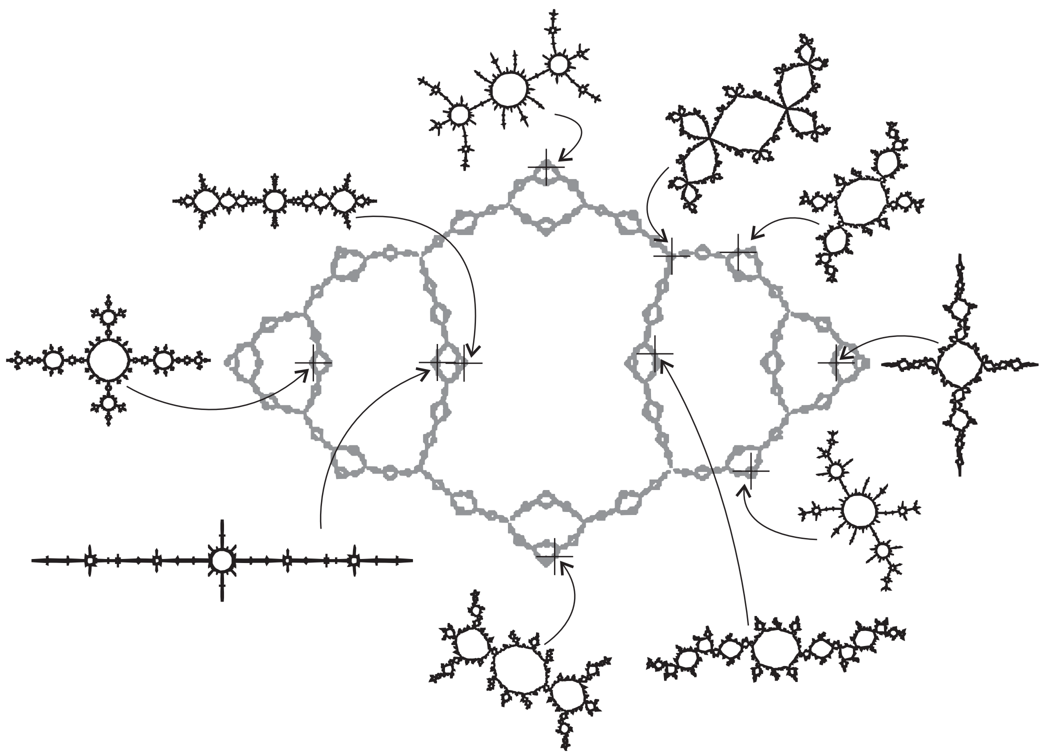

Figure 11: Julia set of F.

We know that $\mathcal{J}_H$ is homeomorphic to the Julia set of $1 - \frac{1}{p^2}$, the limit space $\mathcal{J}_{\widehat{G}}$ is homeomorphic to the Julia set $\mathcal{J}_F$ of the map $F : (z,p) \mapsto \left(1 - \frac{z^2}{p^2}, 1 - \frac{1}{p^2}\right)$. It also follows that the map $\mathcal{J}_{\widehat{G}} \to \mathcal{J}_H$ is in this interpretation just the projection of the Julia set of F onto the second coordinate. Then the fiber of a point $p_0 \in \mathcal{J}_F$ under the projection map is the Julia set of the forward iteration of the quadratic polynomials

$$\mathbb{C} \xrightarrow{1-\frac{z^2}{p_0}} \mathbb{C} \xrightarrow{1-\frac{z^2}{p_1}} \mathbb{C} \xrightarrow{1-\frac{z^2}{p_2}} \cdots,$$

where $p_i = 1 - \frac{1}{p_{i-1}^2}$. This can be used for computer aided visualization of the fibers of the limit space $\mathcal{J}_{\widehat{G}} = \mathcal{J}_F$. See Figure 11 for the Julia set of $1 - \frac{1}{p^2}$ and the fibers of the Julia set of F over the respective points of $\mathcal{J}_{1-1/p^2}$. Recall, that these fibers are also the connected components of $\mathcal{J}_G$.

References

[1] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunić. Branch groups. In *Handbook of Algebra, Vol. 3*, pages 989–1112. North-Holland, Amsterdam, 2003.

[2] Christophe Champetier. L'espace de groupes de type fini. *Topology*, 39:657–680, 2000.

[3] Piotr W. Gawron, Volodymyr V. Nekrashevych, and Vitaly I. Sushchansky. Conjugation in tree automorphism groups. *Int. J. of Algebra and Computation*, 11(5):529–547, 2001.

[4] Yaroslav V. Lavreniuk and Volodymyr V. Nekrashevych. Rigidity of branch groups acting on rooted trees. *Geom. Dedicata*, 89(1):155–175, 2002.

[5] Volodymyr Nekrashevych. *Self-similar groups*, volume 117 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., Providence, RI, 2005.