Self-similar groups and their limit spaces

Volodymyr Nekrashevych

October 13, 2008,
Banff

Self-similar groups

Definition

A *self-similar group* (G, X) is a group G with a faithful action on $\mathsf{X}^* = \{x_1 \ldots x_n \ : \ x_i \in \mathsf{X}\}$ such that for every $g \in G$ and $x \in \mathsf{X}$ there exist $h \in G$ and $y \in \mathsf{X}$ such that

$$g(xw) = yh(w)$$

for all $w \in \mathsf{X}^*$.

If the action is self-similar, then for every $v, w \in \mathsf{X}^*$ and $g \in G$ there exists $g|_v \in G$ such that

$$g(vw) = g(v)g|_v(w).$$

for all $w \in \mathsf{X}^*$.

Example: odometer

Consider the cyclic group generated by the transformation a of $\{0,1\}^*$ given by the recurrent rule

$$a(0w) = 1w, \quad a(1w) = 0a(w).$$

It acts as adding 1 to a dyadic integer:

$$a(x_1x_2\dots x_n) = y_1y_2\dots y_n \Leftrightarrow 1 + \sum_{k=1}^{n} 2^{k-1}x_k = \sum_{k=1}^{n} 2^{k-1}y_k \pmod{2^n}.$$

Example: Grigorchuk group

Consider the group generated by the transformations a, b, c, d of $\{0, 1\}^*$ given by

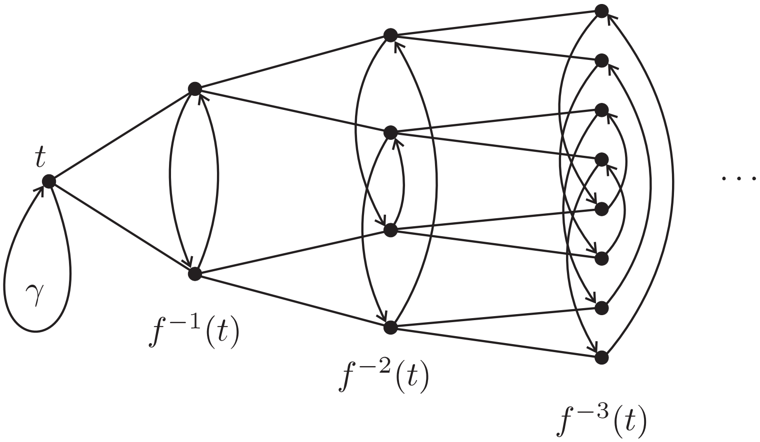
$$\begin{aligned} a(0w) &= 1w, & a(1w) &= 0w, \\ b(0w) &= 0a(w), & b(1w) &= 1c(w), \\ c(0w) &= 0a(w), & c(1w) &= 1d(w), \\ d(0w) &= 0w, & d(1w) &= 1b(w). \end{aligned}$$

Iterated monodromy groups

Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a covering of a topological space by its subset. Choose a basepoint $t \in \mathcal{M}$. We get a *rooted tree of preimages*:

$$T = \{t\} \cup f^{-1}(t) \cup f^{-2}(t) \cup f^{-3}(t) \cup \cdots$$

The fundamental group $\pi_1(\mathcal{M}, t)$ acts on it in the natural way.



The quotient of the action of $\pi_1(\mathcal{M}, t)$ by the kernel of the action is called the *iterated monodromy group* $\mathrm{IMG}(f)$.

There is a natural labeling of vertices of the tree of preimages T by finite words over an alphabet X, $|\mathsf{X}| = \deg f$, such that the action of $\mathrm{IMG}\,(f)$ is self-similar.

For example, the odometer action of $\mathbb{Z}$ is $\mathrm{IMG}\,(z^2)$; the group generated by

$$a(0w) = 1w, \quad a(1w) = 0b(w), \quad b(0w) = 0w, \quad b(1w) = 1a(w)$$

is $\mathrm{IMG}\,(z^2 - 1)$.

Contracting groups

Definition

A self-similar group G is called *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n such that $g|_v \in \mathcal{N}$ whenever $|v| \geq n$.

The smallest set $\mathcal{N}$ satisfying this property is called the *nucleus* of the group.

Contracting groups have solvable word problem (of polynomial complexity). Conjugacy problem?

They have no free subgroups. Are they amenable?

They are typically infinitely presented. (All, except for virtually nilpotent?)

Many are (weakly) branch. (All, except for virtually nilpotent?)

Limit space $\mathcal{J}_G$

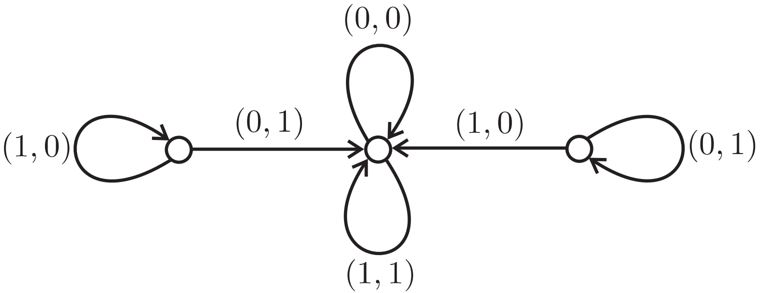
Consider the space $\mathsf{X}^{-\omega}$ of the left-infinite words $\ldots x_2 x_1$.
Fix a self-similar group G. Two sequences $\ldots x_2 x_1, \ldots y_2 y_1$ are equivalent if there exists a finite set $A \subset G$ and a sequences $g_k \in A$ such that

$$g_k(x_k \ldots x_1) = y_k \ldots y_1.$$

for all k.
The quotient of $\mathsf{X}^{-\omega}$ by this equivalence relation is the *limit space* $\mathcal{J}_G$.
The equivalence relation is invariant under the shift $\ldots x_2 x_1 \mapsto \ldots x_3 x_2$, hence the shift induces a continuous map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$.

Proposition

Sequences $\ldots x_2 x_1, \ldots y_2 y_1 \in \mathsf{X}^{-\omega}$ *are equivalent if and only if there exists a path* $\ldots e_2 e_1$ *in the Moore diagram of the nucleus* $\mathcal{N}$ *such that the arrow* e_n *is labeled by* (x_n, y_n)*.*



Elementary properties

The limit space $\mathcal{J}_G$ is metrizable, finite-dimensional, compact.

It is connected if the group G is level-transitive.

It is locally connected if the group G is self-replicating, i.e., if for every $x, y \in \mathsf{X}$ and $h \in G$ there exists $g \in G$ such that $g(x) = y$ and $g|_x = h$.

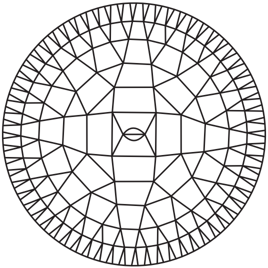
Julia sets and limit spaces

If $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is an *expanding* partial self-covering, then $\mathrm{IMG}(f)$ is contracting and $(\mathcal{J}_{\mathrm{IMG}(f)}, \mathsf{s})$ is topologically conjugate to $(\mathcal{J}_f, f)$, where $\mathcal{J}_f$ is the set of accumulation points of the set $\bigcup_{k\geq 1} f^{-k}(t)$.

We get in this way a symbolic presentation of the action of f on its Julia set.

Limit spaces as Gromov boundaries

Let a contracting group G be generated by a finite set S. Consider the graph with the set of vertices X^* where a vertex v is connected to $s(v)$ for $s \in S$ and to xv for $x \in \mathsf{X}$.



This graph is Gromov hyperbolic and its boundary is homeomorphic to $\mathcal{J}_G$.

Limit solenoid

Consider the space $\mathsf{X}^{\mathbb{Z}}$ of bi-infinite sequences

$$\ldots x_{-2}x_{-1}x_0 \,.\, x_1x_2\ldots.$$

Let (G, X) be a contracting group. Two sequences $\ldots x_{-1}x_0 \,.\, x_1x_2\ldots, \ldots y_{-1}y_0 \,.\, y_1y_2\ldots$ are equivalent (with respect to the action of G) if there exists a finite set $A \subset G$ and a sequence $g_k \in A$ such that

$$g_k(x_kx_{k+1}\ldots) = y_ky_{k+1}\ldots$$

for all $k \in \mathbb{Z}$.
The quotient of $\mathsf{X}^{\mathbb{Z}}$ by the equivalence relation is called the *limit solenoid* $\mathcal{S}_G$ of the group (G, X).

The limit solenoid is a compact metrizable space. The shift on $\mathsf{X}^{\mathbb{Z}}$ induces a homeomorphism of $\mathcal{S}_G$.

The limit solenoid is connected if G is level-transitive.

A *leaf* of the solenoid $\mathcal{S}_G$ is the set of points represented by sequences $\ldots x_{-2}x_{-1}x_0 \,.\, x_1x_2\ldots$ such that $x_1x_2\ldots$ belongs to an orbit of the action of G on X^{ω}.

It follows from the definition of the equivalence relation on $\mathsf{X}^{\mathbb{Z}}$ that the leaves are disjoint. If the action is self-replicating, then the leaves are mapped by the shift to leaves.

Examples

Let f be a post-critically finite complex rational function. Then the limit solenoid of $\mathrm{IMG}(f)$ is the lift of the Julia set of f to the inverse limit of the sequence

$$\widehat{\mathbb{C}} \stackrel{f}{\longleftarrow} \widehat{\mathbb{C}} \stackrel{f}{\longleftarrow} \widehat{\mathbb{C}} \stackrel{f}{\longleftarrow} \cdots .$$

Examples

Let $(G, \mathsf{X}) = (\mathbb{Z}, \{0, 1\})$ be the binary odometer action. Then the limit solenoid is the space of binary sequences $\ldots x_{-1}x_0 \,.\, x_1x_2 \ldots$ modulo the equivalence relation identifying two sequences $\ldots x_{-1}x_0 \,.\, x_1x_2 \ldots$ and $\ldots y_{-1}y_0 \,.\, y_1y_2 \ldots$ iff

$$\sum_{k=0}^{\infty} 2^{-k}x_{-k} - \sum_{k=0}^{\infty} 2^{-k}y_{-k} = \sum_{k=1}^{\infty} 2^{k}x_{k} - \sum_{k=1}^{\infty} 2^{k}y_{k},$$

where both differences belong to $\mathbb{Z}$.

It follows that the limit solenoid of the binary odometer is the inverse limit of the circle $\mathbb{R}/\mathbb{Z}$ with respect to the double self-coverings $x \mapsto 2x$.

More generally, the limit solenoid $\mathcal{S}_G$ of a contracting group is the inverse limit of the sequence

$$\mathcal{J}_G \xleftarrow{\mathsf{s}} \mathcal{J}_G \xleftarrow{\mathsf{s}} \cdots .$$

Examples

Let $(\mathbb{Z}^n, \mathsf{X})$ be a self-replicating free abelian group. Then there exists a matrix $A \in M_{n\times n}(\mathbb{Z})$ such that $\det(A) = |\mathsf{X}|$, and a coset transversal $\{r_0, \ldots, r_{d-1}\}$ of $\mathbb{Z}^n$ modulo $A\mathbb{Z}^n$ such that the action of $\mathbb{Z}^n$ on X^ω describes the natural action of $\mathbb{Z}^n$ on the formal series

$$r_{x_0} + Ar_{x_1} + A^2 r_{x_2} + \cdots .$$

The group $(\mathbb{Z}^n, \mathsf{X})$ is contracting if and only if all eigenvalues of A are greater than one. The limit space of $(\mathbb{Z}^n, \mathsf{X})$ will be the torus $\mathbb{R}^n/\mathbb{Z}^n$, where a sequence $\ldots x_2 x_1$ encodes the point

$$A^{-1} r_{x_1} + A^{-2} r_{x_2} + A^{-3} r_{x_3} + \cdots .$$

The limit solenoid is the space of all formal series

$$\sum_{k=-\infty}^{+\infty} A^k r_{i_k}.$$

Tiles

Let (G, X) be a contracting group and let $\mathcal{S}_G$ be its solenoid.
A *tile* $\mathcal{T}_v$ for $v \in \mathsf{X}^\omega$ is the set of points of $\mathcal{S}_G$ represented by the sequences of the form $\ldots x_{-1}x_0 \,.\, v$.
Tiles $\mathcal{T}_{v_1}$ and $\mathcal{T}_{v_2}$ intersect if and only if there exists an element g of the nucleus of G such that $g(v_1) = v_2$.
It follows directly from the definition that

$$\mathsf{s}(\mathcal{T}_v) = \bigcup_{x \in \mathsf{X}} \mathcal{T}_{xv}.$$

Every leaf is a union of tiles. We consider the leaves with the *inductive limit topology* defined with respect to this decomposition into the union of tiles.

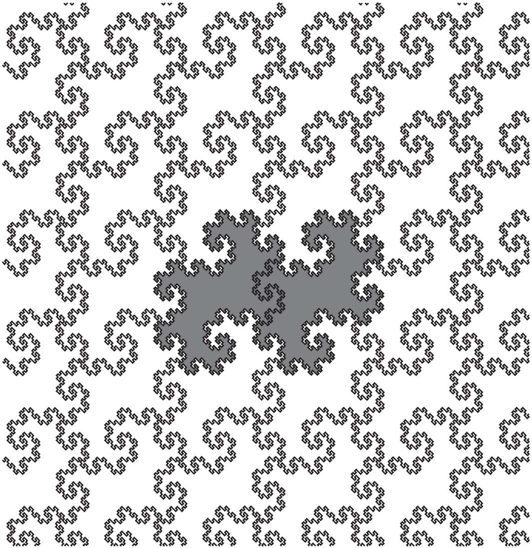
If for every $g \in G$ there exists $v \in \mathsf{X}^*$ such that $g|_v = 1$, then every tile is closure of its interior (in the inductive limit topology of the leaf) and different tiles have disjoint interiors.

Let $(\mathbb{Z}^n, \mathsf{X})$ be a contracting self-replicating action of $\mathbb{Z}^n$. Let L_v be the leaf of points represented by sequences $\ldots x_{-1}x_0 \,.\, g(v)$ for $g \in \mathbb{Z}^n$. The action of $\mathbb{Z}^n$ on X^ω is free, hence g is uniquely defined by $g(v)$ and v. We can identify then the leaf L_v with $\mathbb{R}^n$ by

$$[\ldots x_{-1}x_0 \,.\, g(v)] \mapsto g + \sum_{k=0}^{\infty} A^{-k} r_{x_{-k}}.$$

The tile $\mathcal{T}_v$ is identified then with the set of sums

$$\sum_{k=0}^{\infty} A^{-k} r_{x_{-k}}.$$



"Iterated monodromy group" of the Penrose tiling

$$
\begin{aligned}
S(aw) &= cw \\
S(bw) &= b \cdot M(w) \\
S(cw) &= aw \\
\\
M(aw) &= a \cdot L(w) \\
M(bw) &= cw \\
M(caw) &= c \cdot M(aw) \\
M(cbw) &= bbw \\
M(ccw) &= bcw
\end{aligned}
$$

$$
\begin{aligned}
L(aaw) &= b \cdot S(aw) \\
L(abw) &= a \cdot M(bw) \\
L(acw) &= a \cdot M(cw) \\
L(bbw) &= b \cdot S(bw) \\
L(bcw) &= a \cdot S(cw) \\
L(cw) &= c \cdot L(w)
\end{aligned}
$$