Contracting Self-Similar Groups

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Self-similar groups

Definition

A *self-similar group* is a group G with a faithful action on $\mathsf{X}^* = \{x_1 \ldots x_n \ : \ x_i \in \mathsf{X}\}$ such that for every $g \in G$ and $x \in \mathsf{X}$ there exist $h \in G$ and $y \in \mathsf{X}$ such that

$$g(xw) = yh(w)$$

for all $w \in \mathsf{X}^*$.

$$g \cdot x = y \cdot h$$

$$x : w \mapsto xw$$

For every $g \in G$ and $v \in \mathsf{X}^*$ there exists $g_v \in G$ such that

$$g(vw) = g(v) \cdot g_v(w)$$

$$g \cdot v = g(v) \cdot g_v$$

Definition

A self-similar group (G, X) is *contracting* if there exists a finite subset $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that

$$g_v \in \mathcal{N}$$

for all $v \in \mathsf{X}^*$, $|v| \geq n$.

The minimal subset $\mathcal{N}$ satisfying the definition is called the *nucleus*.

Contracting groups have word problem of polynomial complexity.

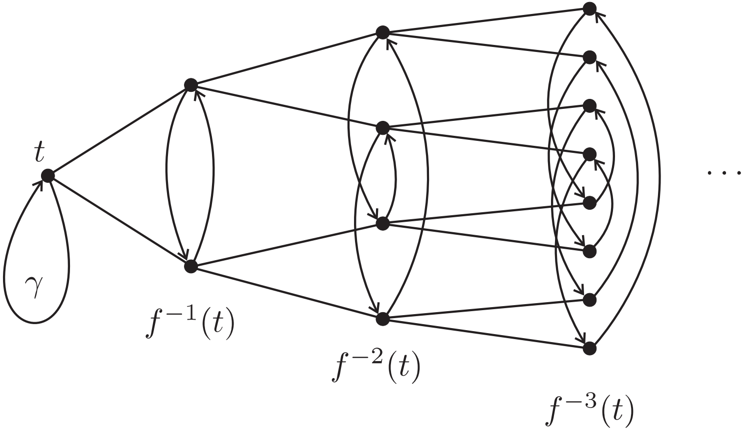
Many are infinitely presented. (All except for some virtually nilpotent?)

Many are amenable. (All?)

Many groups of intermediate growth are contracting.

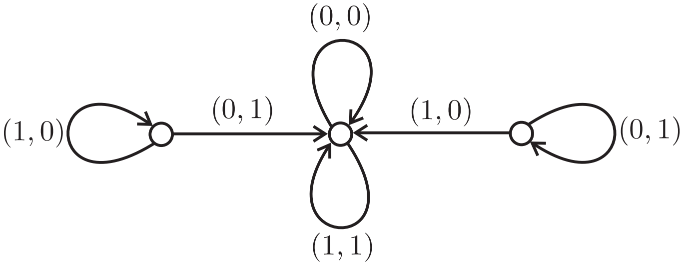
Iterated monodromy groups

$f : \mathcal{M}_1 \longrightarrow \mathcal{M}$



Iterated monodromy group of z^2 is $\mathbb{Z}$ generated by

$$a(0w) = 1w, \qquad a(1w) = 0a(w).$$



Iterated monodromy group of $z^2 - 1$ is generated by two transformations a, b given by

$$a(0w) = 0w, \quad a(1w) = 1b(w)$$
$$b(0w) = 1w, \quad b(1w) = 0a(w).$$

Limit space $\mathcal{J}_G$

Consider the space $\mathsf{X}^{-\omega}$ of the left-infinite words $\ldots x_2x_1$.
Fix a contracting group G. Two sequences $\ldots x_2x_1, \ldots y_2y_1$ are equivalent if there exists a finite set $A \subset G$ and a sequence $g_k \in A$ such that

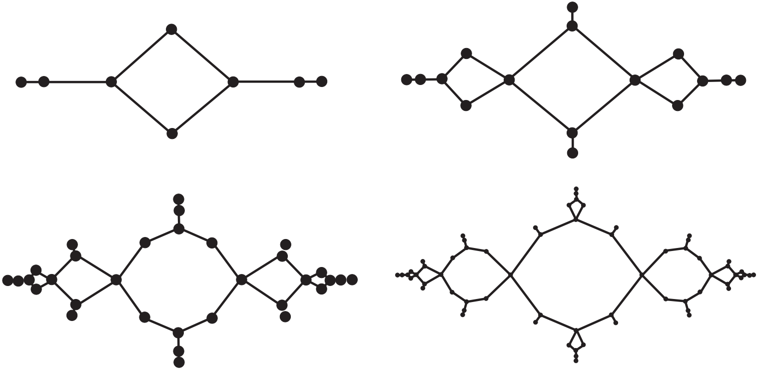
$$g_k(x_k \ldots x_1) = y_k \ldots y_1.$$

for all k.
The quotient of $\mathsf{X}^{-\omega}$ by this equivalence relation is the *limit space* $\mathcal{J}_G$.
The equivalence relation is invariant under the shift

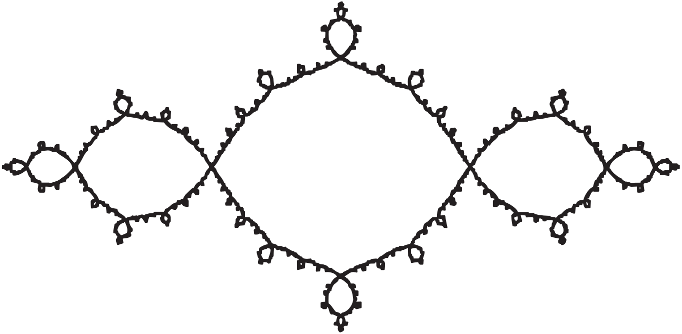
$$\ldots x_2x_1 \mapsto \ldots x_3x_2,$$

hence the shift induces a continuous map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$.

Approximation by Schreier graphs



Approximation by Schreier graphs

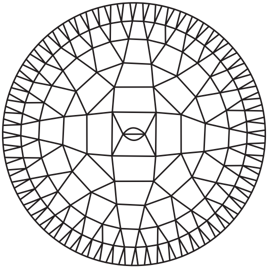


Limit spaces as Gromov boundaries

Let $\langle G, \mathsf{X} \rangle$ be the semigroup of transformations of X^* generated by

$$g : w \mapsto g(w), \qquad x : w \mapsto xw.$$

Let Γ be the *left* Cayley graph of $\langle G, \mathsf{X} \rangle$. G acts on Γ from the *right*. If G is contracting, then Γ / G is Gromov hyperbolic with boundary homeomorphic to $\mathcal{J}_G$.



Functoriality

A *morphism* $(G_1, \mathsf{X}_1) \longrightarrow (G_2, \mathsf{X}_2)$ of self-similar groups is a semigroup homomorphism $F : \langle G_1, \mathsf{X}_1\rangle \longrightarrow \langle G_2, \mathsf{X}_2\rangle$ such that $F(G_1) \subseteq G_2$ and $F(\mathsf{X}_1 \cdot G_1) \subseteq \mathsf{X}_2 \cdot G_2$.

Contraction and the limit dynamical system depend only on the isomorphism class of (G, X) (in the category of self-similar groups). The limit dynamical system is a functor from the category of contracting self-similar groups to the category of dynamical systems (and semiconjugacies).

In particular, if H is a self-similar subgroup of a contracting self-similar group G, then the embedding $H \hookrightarrow G$ induces a surjective continuous map $\mathcal{J}_H \longrightarrow \mathcal{J}_G$.

Iterated Monodromy Groups

Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a covering by a subset. Choose a basepoint t and let $\mathcal{F}$ be the set of all paths from t to points of

$$T_f = \bigsqcup_{n \geq 0} f^{-n}(t).$$

Every $\ell \in \mathcal{F}$ defines a transformation of T_f mapping $z \in f^{-n}(t)$ to the end of the f^n-preimage of ℓ starting at z.

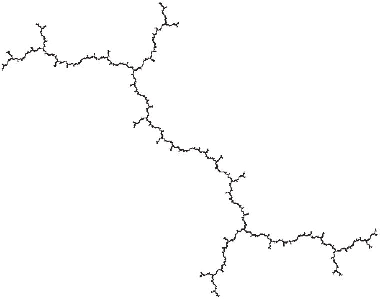
The obtained semigroup of transformations of T_f is $\langle \mathrm{IMG}(f), \mathsf{X} \rangle$ for a self-similar action of $\mathrm{IMG}(f)$.

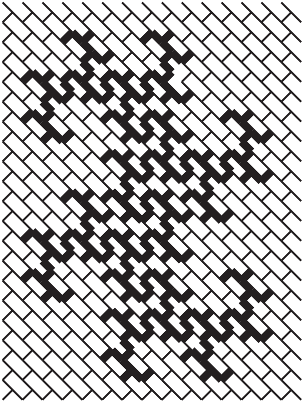
Julia sets and limit spaces

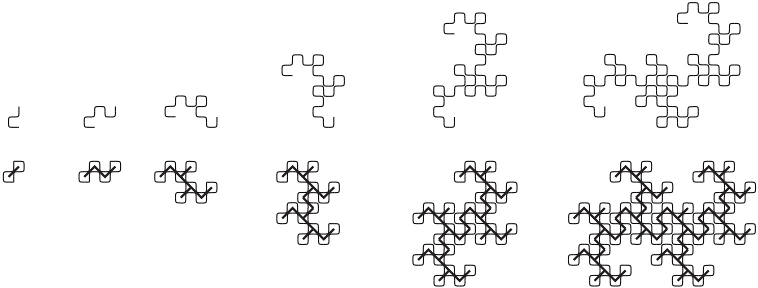
If $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is an *expanding* partial self-covering, then $(\mathrm{IMG}(f), \mathsf{X})$ is contracting and $(\mathcal{J}_{\mathrm{IMG}(f)}, \mathsf{s})$ is topologically conjugate to $(\mathcal{J}_f, f)$, where $\mathcal{J}_f$ is the set of accumulation points of $\bigsqcup_{n\geq 0} f^{-n}(t)$.

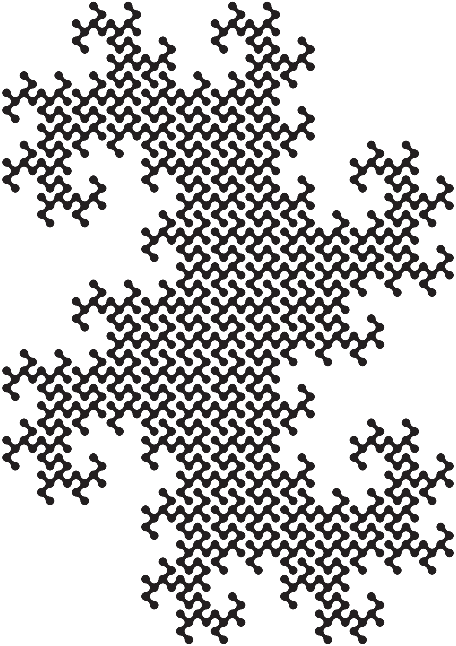
The iterated monodromy group of $(\mathcal{J}_{\mathrm{IMG}(f)}, \mathsf{s})$ is $(\mathrm{IMG}(f), \mathsf{s})$.

Mate $f(z) = z^2 - 0.2282\ldots + 1.1151\ldots i$ with itself. The obtained embedding of $\mathrm{IMG}(f)$ into a larger group induces a plane-filling dendrite, described by J. Milnor.









An axiomatic definition

Let (G, X) be a self-similar group and let $\mathcal{X}$ be a right G-space. Let $\mathcal{X} \otimes \mathsf{X} \cdot G$ be the quotient of $\mathcal{X} \times \mathsf{X} \cdot G$ by

$$\xi \otimes g \cdot m = \xi \cdot g \otimes m.$$

$\mathcal{X} \otimes \mathfrak{M}$ is a right G-space with respect to

$$(\xi \otimes m) \cdot g = \xi \otimes (m \cdot g).$$

A *self-similarity* is a homeomorphism $\Phi : \mathcal{X} \otimes \mathsf{X} \cdot G \longrightarrow \mathcal{X}$ such that

$$\Phi(\xi \cdot g) = \Phi(\xi) \cdot g$$

for all $\xi \in \mathcal{X} \otimes \mathsf{X} \cdot G$ and $g \in G$.
We get then a collection of continuous maps $\xi \mapsto \xi \otimes x$ for $x \in \mathsf{X}$ such that

$$(\xi \cdot g) \otimes x = (\xi \otimes y) \cdot h$$

for $y = g(x)$ and $h = g_x$, i.e., when $g(xw) = yh(w)$ for all w.

Let $\mathcal{X}$ be a proper, co-compact, locally compact, metrizable right G-space. A relation $R \subset \mathcal{X} \times \mathcal{X}$ is *bounded* if there exists a compact set $K \subset \mathcal{X} \times \mathcal{X}$ such that $R \subset \bigcup_{g \in G} K \cdot g$.

A neighborhood of the diagonal $U \subset \mathcal{X} \times \mathcal{X}$ is *uniform* if it contains a G-invariant open neighborhood of the diagonal.

Theorem

Let (G, X) be a contracting group. Then there exists a right G-space $\mathcal{X}_G$ and a contracting self-similarity $\Phi : \mathcal{X} \otimes \mathsf{X} \cdot G \longrightarrow \mathcal{X}$, i.e., such that for any uniform neighborhood U of the diagonal and for any bounded relation R there exists n such that $(\xi_1 \otimes v, \xi_2 \otimes v) \in U$ for all $(\xi_1, \xi_2) \in R$ and all $v \in \mathsf{X}^m$ for $m \geq n$.

Moreover, $\mathcal{X}$ and the self-similarity are unique: if $\mathcal{X}'$ is another space with a contracting self-similarity, then there exists a homeomorphism $F : \mathcal{X} \longrightarrow \mathcal{X}'$ such that

$$F(\xi \cdot g) = F(\xi) \cdot g, \quad F(\xi \otimes x) = F(\xi) \otimes x$$

for all $\xi \in \mathcal{X}$, $g \in G$ and $x \in \mathsf{X}$.

The unique G-space $\mathcal{X}$ is called the *limit G-space*. The orbispace $\mathcal{X}/G$ is homeomorphic to $\mathcal{J}_G$.

Literature

1. John Milnor, *Pasting together Julia sets: a worked out example of mating*, Experiment. Math. **13** (2004), no. 1, 55–92.
2. Laurent Bartholdi and Volodymyr V. Nekrashevych, *Thurston equivalence of topological polynomials*, Acta Math. **197** (2006), no. 1, 1–51.
3. Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, Amer. Math. Soc., Providence, RI, 2005.
4. Volodymyr Nekrashevych, *Symbolic dynamics and self-similar groups*, preprint, available at `http:/www.math.tamu.edu/ nekrash/Preprints/filling.pdf`