Self-similar and branch groups

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Groups acting on rooted trees

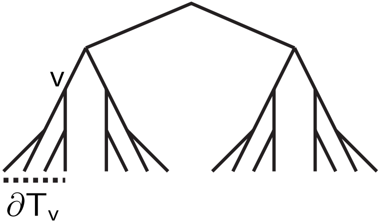
A *rooted tree* is a tree T with a fixed vertex v_0 called the *root*. (All our trees are locally finite and do not have vertices of degree 1.)

An *automorphism* of a rooted tree T is an automorphism of the tree T that fixes the root.

A *level number* n is the set of vertices on distance n from v_0. A group $G \leq \mathrm{Aut}(T)$ is *level-transitive* if it is transitive on every level. T is *level-transitive* if $\mathrm{Aut}(T)$ is level-transitive.

Boundary of a rooted tree

The *boundary* ∂T is the set of simple paths in T starting at the root. Basis of topology of ∂T consists of sets $\partial T_v \subset \partial T$ of paths passing through a vertex $v \in T$.



If T is level-transitive, then we also have a measure μ on ∂T given by

$$\mu(\partial T_v) = \frac{1}{|L_n|},$$

where $v \in L_n$. $\mathrm{Aut}(T)$ acts on ∂T by (measure preserving) homeomorphisms.

Stabilizers

Let $G \leq \mathrm{Aut}(T)$. For $v \in T$ denote by G_v the stabilizer of v. The *nth level stabilizer* G_n is $\bigcap_{v \in L_n} G_v$, where L_n is the nth level.

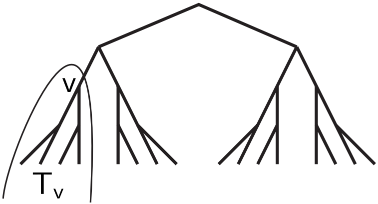
G_v and G_n are subgroups of finite index, and $\bigcap_n G_n = \{1\}$, hence G is residually finite.

For any H and a decreasing sequence of finite index subgroups H_n such that $\bigcap H_n = \{1\}$ we have $H \hookrightarrow \mathrm{Aut}(T)$ for

$$T = \bigsqcup H/H_n$$

of cosets.

The *rigid stabilizer* $G[v]$ is the set of automorphisms acting trivially outside of T_v.



The *nth level rigid stabilizer* is $\mathrm{Rist}_n = \prod_{v \in L_n} G[v]$.
A level-transitive group $G \in \mathrm{Aut}(T)$ is *weakly branch* if Rist_n are non-trivial (equivalently, infinite).
It is *branch* if Rist_n are of finite index.

Schreier graphs

Let $G \leq \mathrm{Aut}(T)$ be generated by finite set S. Denote by $\Gamma_n(G, S)$ the graph with the vertex set L_n in which two vertices are connected by an edge if they are v and $s(v)$ for $s \in S$.

For $w \in \partial T$ define $\Gamma_w(G, S)$ as the graph with the set of vertices $G(w)$ with the same adjacency rule.

Tree of words

Let X be a finite alphabet. The tree rooted tree X^* is the free monoid generated by X were v is connected to vx for $v \in X^*$ and $x \in X$, i.e., it is the right Cayley graph of the monoid.

The nth level is then X^n. The boundary is naturally identified with the space X^ω of right-infinite sequences with the product topology and the product μ of uniform distributions. Automorphisms of X^* are interpreted as *automata-transducers*, since beginning of length $|v|$ of $g(vw)$ does not depend on w.

Self-similar groups

For every $g \in \mathrm{Aut}(\mathsf{X}^*)$ and $v \in \mathsf{X}^*$ there exists $h \in \mathrm{Aut}(\mathsf{X}^*)$ such that

$$g(vw) = g(v)h(w)$$

for all $w \in \mathsf{X}^*$. We denote $h = g|_v$.

Definition

A group $G \leq \mathrm{Aut}(\mathsf{X}^*)$ is *self-similar* if for all $g \in G$ and $x \in \mathsf{X}$ we have

$$g|_x \in G.$$

Self-similar groups can be interpreted as automata. When in state $g \in G$ it reads a letter $x \in \mathsf{X}$ then it goes to state $g|_x$ and gives $g(x)$ on output:

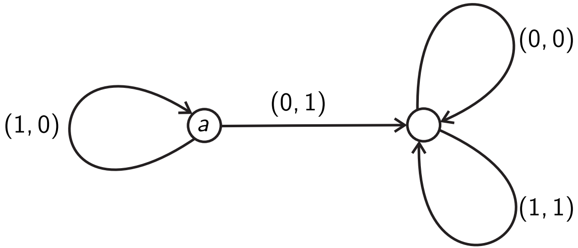
$$g(xw) = g(x)g|_x(w).$$

Examples

Consider $a \in \mathrm{Aut}(\mathsf{X}^*)$ for $\mathsf{X} = \{0, 1\}$ defined by

$$a(0w) = 1w, \qquad a(1w) = 0a(w).$$

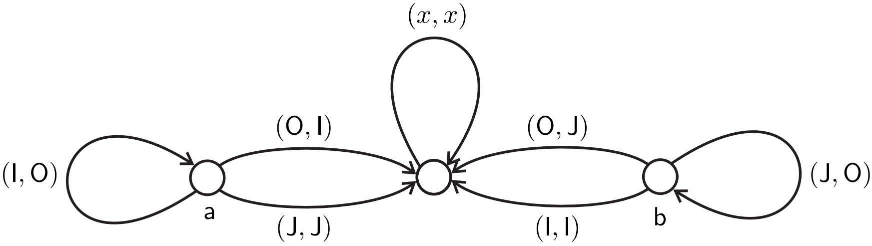
This transformation (of X^* and of X^ω) is called the *(binary) adding machine*. It generates a self-similar action of $\mathbb{Z}$.



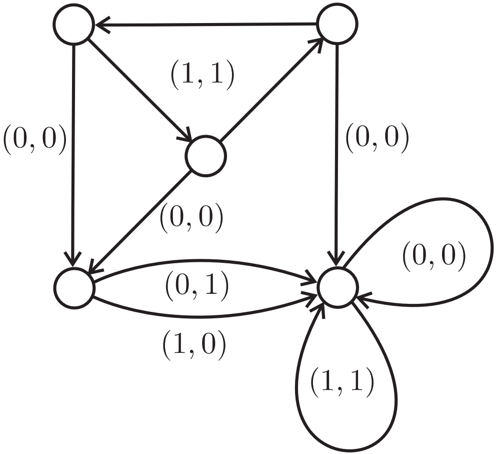
"Interlaced" adding machines

Consider $a, b \in \mathrm{Aut}(\mathsf{X}^*)$ for $\mathsf{X} = \{O, I, J\}$ defined by

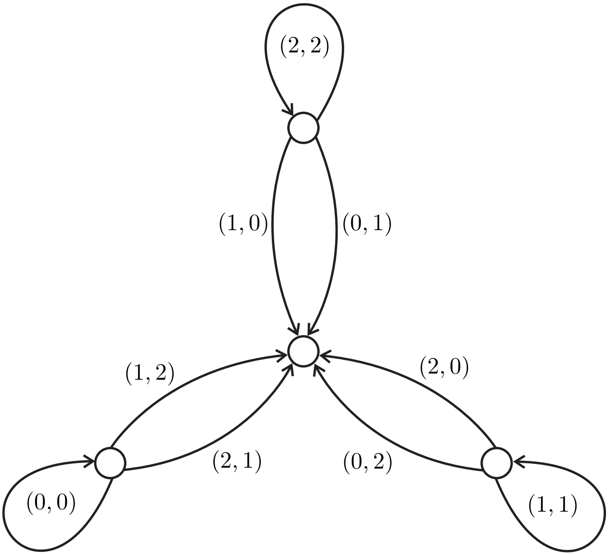
$$a(Ow) = Iw, \qquad a(Iw) = Oa(w), \qquad a(Jw) = Jw,$$
$$b(Ow) = Jw, \qquad b(Iw) = Iw, \qquad b(Jw) = Ob(w).$$



Grigorchuk group



Hanoi towers group



Schreier graphs of self-similar groups

For any f.g. group $G \leq \mathrm{Aut}(\mathsf{X}^*)$ the map $vx \mapsto v$ induces a covering

$$\Gamma_{n+1}(G, S) \longrightarrow \Gamma_n(G, S).$$

The inverse limit w.r.t. these maps is the profinite graph of the action of G on X^ω. Its connected components are the Schreier graphs $\Gamma_w(G, S)$ for $w \in \mathsf{X}^\omega$.

If S is a *self-similar* generating set, (i.e., if $s|_x \in S$ for all $s \in S$ and $x \in \mathsf{X}$) then $xv \mapsto v$ induces a map

$$\Gamma_{n+1}(G, S) \longrightarrow \Gamma_n(G, S).$$

Examples

1. **Adding machine.** The Schreier graphs $\Gamma_n(G, \{a\})$ are cycles of length 2^n. The maps $vx \mapsto v$ are the natural coverings. The maps $xv \mapsto v$ collapses every other edge.
2. **Interlaced adding machines**