Self-similar groups and hyperbolic groupoids III

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Pseudogroups

A *pseudogroup* of local homeomorphisms of a space $\mathcal{X}$ is a collection $\widetilde{\mathfrak{G}}$ of homeomorphisms $F : U \longrightarrow V$ between open subsets of $\mathcal{X}$ closed under taking

- compositions;
- inverses;
- restrictions onto open subsets;
- unions: if $F : U \longrightarrow V$ is a homeomorphisms such that for a covering $\{U_i\}$ of U we have $F|_{U_i} \in \widetilde{\mathfrak{G}}$, then $F \in \widetilde{\mathfrak{G}}$).

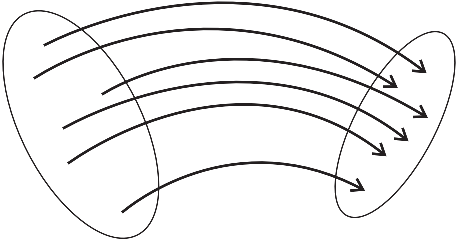
We assume that $Id : \mathcal{X} \longrightarrow \mathcal{X}$ belongs to $\widetilde{\mathfrak{G}}$.

Equivalence

Two pseudogroups $(\widetilde{\mathfrak{G}}_1, \mathcal{X}_1)$ and $(\widetilde{\mathfrak{G}}_2, \mathcal{X}_2)$ are *equivalent* if there exists a pseudogroup $(\widetilde{\mathfrak{G}}, \mathcal{X}_1 \sqcup \mathcal{X}_2)$ such that restriction of $\widetilde{\mathfrak{G}}$ onto $\mathcal{X}_i$ is $\widetilde{\mathfrak{G}}_i$, and every $\widetilde{\mathfrak{G}}$-orbit is a union of a $\widetilde{\mathfrak{G}}_1$-orbit and a $\widetilde{\mathfrak{G}}_2$-orbit.

Groupoids of germs

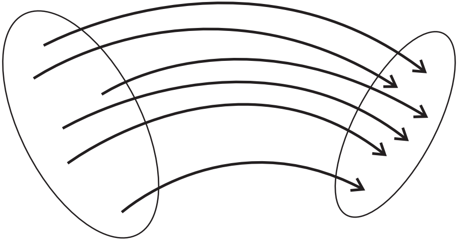
Let $(\widetilde{\mathfrak{G}}, \mathcal{X})$ be a pseudogroup. A *germ* is an equivalence class of (F, x), $F \in \widetilde{\mathfrak{G}}$, $x \in \mathrm{Dom}(F)$, where $(F_1, x) = (F_2, x)$ if there is a neighborhood U of x such that $F_1|_U = F_2|_U$.
The set of all germs of $\widetilde{\mathfrak{G}}$ has a natural topology, where the set of all germs of an element $F \in \widetilde{\mathfrak{G}}$ is declared to be open.



Groupoids of germs

The space $\mathfrak{G}$ of all germs of a pseudogroup $\widetilde{\mathfrak{G}}$ is a *groupoid*: we can multiply the germs and take inverses. We denote $\mathsf{o}(F, x) = x$ and $\mathsf{t}(F, x) = F(x)$ (*origin* and *target*).

The pseudogroup $\widetilde{\mathfrak{G}}$ is uniquely determined by its topological groupoid of germs $\mathfrak{G}$. It is the pseudogroup of *bisections* of $\mathfrak{G}$. An open subset $U \subset \mathfrak{G}$ is a bisection if $\mathsf{o} : U \longrightarrow \mathsf{o}(U)$ and $\mathsf{t} : U \longrightarrow \mathsf{t}(U)$ are homeomorphisms. Any bisection U defines a local homeomorphism $\mathsf{o}(g) \mapsto \mathsf{t}(g), g \in U$.



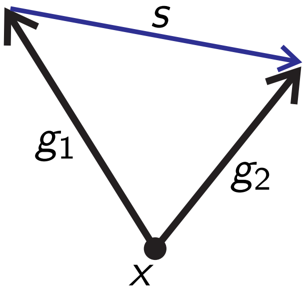
Compactly generated groupoids

Definition

Let $(\mathfrak{G}, \mathcal{X})$ be a groupoid of germs. $(S, \mathcal{X}_1)$, where $S \subset \mathfrak{G}$, $\mathcal{X}_1 \subset \mathcal{X}$ are compact, is a *compact generating pair* if $\mathcal{X}_1$ contains an open set intersecting every $\mathfrak{G}$-orbit, and for every $g \in \mathfrak{G}|_{\mathcal{X}_1}$ there exists n such that $\bigcup_{0\le k\le n}(S \cup S^{-1})^k$ is a neighborhood of g in $\mathfrak{G}|_{\mathcal{X}_1}$.

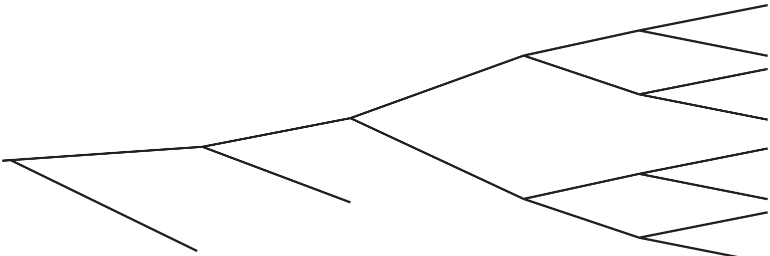
Cayley graphs

If $(S, \mathcal{X}_1)$ is a compact generating pair, and $x \in \mathcal{X}_1$, then the *Cayley graph* $\mathfrak{G}(x, S)$ is the oriented graph with the set of vertices equal to the set of germs $g \in \mathfrak{G}|_{\mathcal{X}_1}$ starting at x, where g_1 is connected to g_2 if there exists $s \in S$ such that $g_2 = sg_1$.



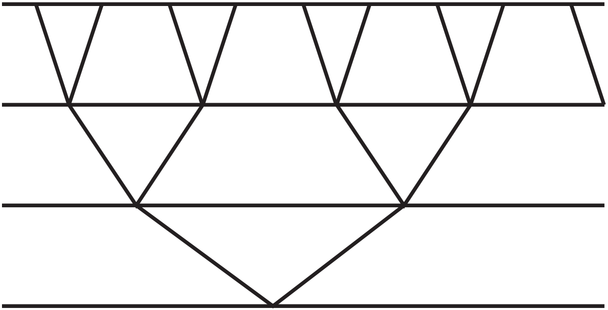
Examples

1. Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be a self-covering of a compact space. Then f is a local homeomorphism, hence its restrictions generate a pseudogroup $\mathfrak{F}$. The set of germs of f and $\mathcal{X}$ form a compact generating pair. The Cayley graphs are regular trees of degree $|\deg f| + 1$.



2. Let (G, X) be a self-similar group. Consider its action on the boundary X^ω of the tree X^*. Let $\widetilde{\mathfrak{G}}$ be the pseudogroup generated by this action and the shift $xw \mapsto w$. Let $\mathfrak{G}$ be the associated groupoid of germs. It is the groupoid of germs of the transformations of the form
$T_{v_2} g T_{v_1}^* : v_1 w \mapsto v_2 g(w)$ for $v_1, v_2 \in \mathsf{X}^*$ and $g \in G$.
Let S be a finite generating set of G. Then the union S_1 of the set of germs of elements of S and of the shift is a compact generating set of $\mathfrak{G}$.

Suppose that (G, X) is *self-replicating*, i.e., the left action of G on $\mathfrak{M} = \mathsf{X} \cdot G$ is transitive. Then the Cayley graph $\mathfrak{G}(x_1 x_2 \ldots, S_1)$ of $\mathfrak{G}$ are generically disjoint unions of Schreier graphs of G-orbits of points $y_1 \ldots y_n x_m x_{m+1} \ldots \in \mathsf{X}^\omega$ connected with each other by edges of the form (w, xw).

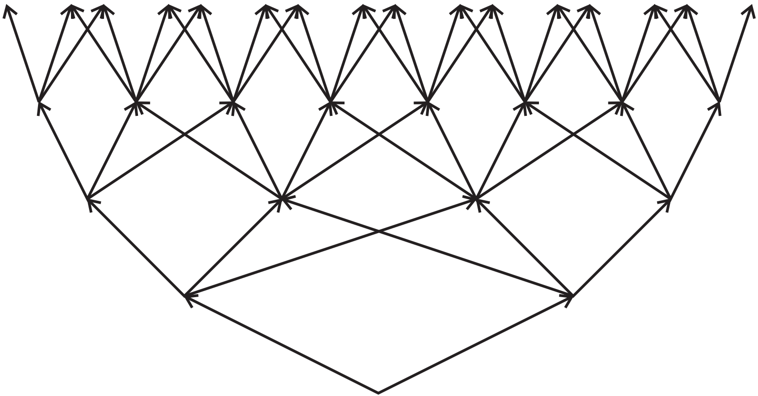


Hyperbolic groupoids

Definition

A groupoid of germs $\mathfrak{G}$ is *hyperbolic* if it is Hausdorff, and there exist a compact generating pair $(S, \mathcal{X}_1)$ and a metric on a neighborhood of $\mathcal{X}_1$ such that

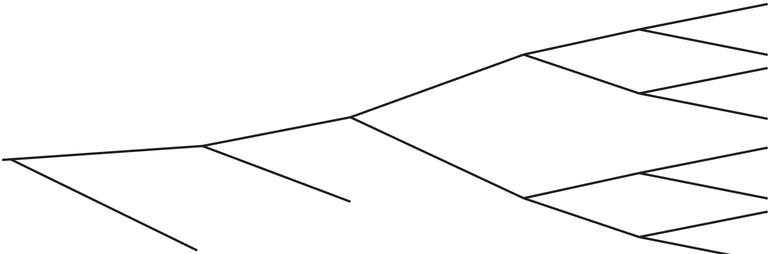
1. The elements of the pseudogroup $\widetilde{\mathfrak{G}}$ are locally Lipschitz.
2. $\mathsf{o}(S) = \mathsf{t}(S) = \mathcal{X}_1$.
3. The elements of S are germs of contractions.
4. The Cayley graphs $\mathfrak{G}(x, S)$ are δ-hyperbolic for all $x \in \mathcal{X}_1$ and some fixed δ.
5. For every $x \in \mathcal{X}_1$ there exists a point $\omega_x \in \partial\mathfrak{G}(x, S)$ such that every direct path of $\mathfrak{G}(x, S^{-1})$ is a quasi-geodesic path converging to ω_x.



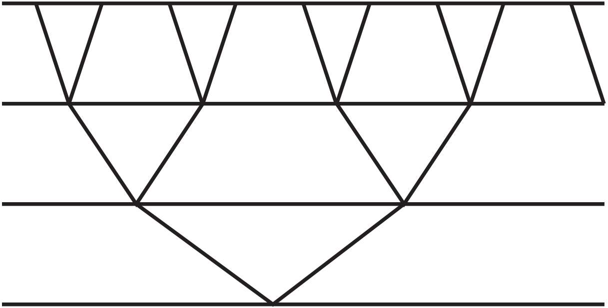
ω_x

Examples

1. Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be an expanding self-covering of a compact space $\mathcal{X}$. Let $\mathfrak{F}$ be the groupoid of germs generated by f. It is hyperbolic.



2. Let (G, X) be a contracting self-similar group. Its groupoid of germs on X^ω is Hausdorff iff $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ is a self-covering of an orbispace, in particular, if it is regular. Then the groupoid of germs generated by G and the shift is hyperbolic.

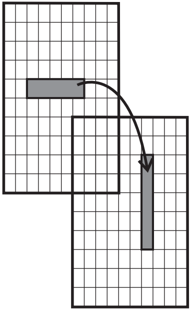


3. Let G be a non-elementary Gromov-hyperbolic group. The groupoid of germs of the action of G on its boundary ∂G is hyperbolic.

4. The groupoid generated by a one-sided shift of finite type is hyperbolic.

5. Let θ be a Pisot number (a real algebraic integer greater than one, such that all its conjugates are less than one in absolute value). Then the groupoid generated by $x \mapsto x + 1$ and $x \mapsto \theta x$ on $\mathbb{R}$ is hyperbolic.

6. Ruelle groupoids of Smale spaces.



The groupoid of germs of the pseudogroups generated by holonomies and the action of f on the stable (resp. unstable) leaves is hyperbolic.