Self-similar groups and hyperbolic groupoids IV

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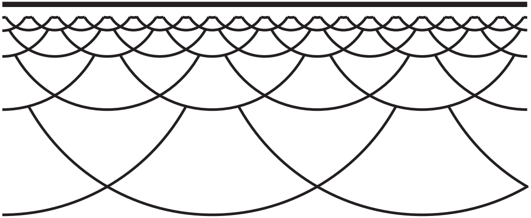
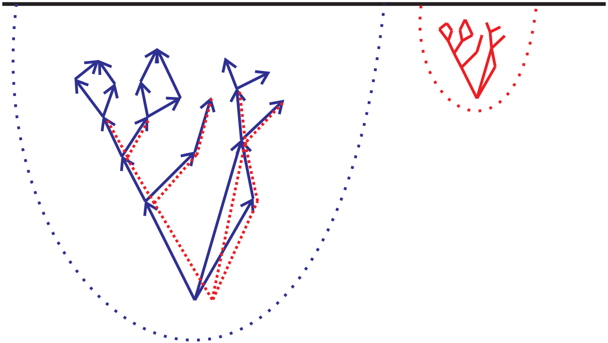
Dual groupoid

Let $\mathfrak{G}$ be a minimal hyperbolic groupoid, and let $x \in \mathfrak{G}^{(0)}$. Denote $\partial\mathfrak{G}_x = \partial\mathfrak{G}(x, S) \setminus \{\omega_x\}$. Let $\overline{\mathfrak{G}_x^{\mathcal{X}_1}}$ be the natural completion of the set of vertices $\mathfrak{G}_x^{\mathcal{X}_1}$ of $\mathfrak{G}(x, S)$ by $\partial\mathfrak{G}_x$.

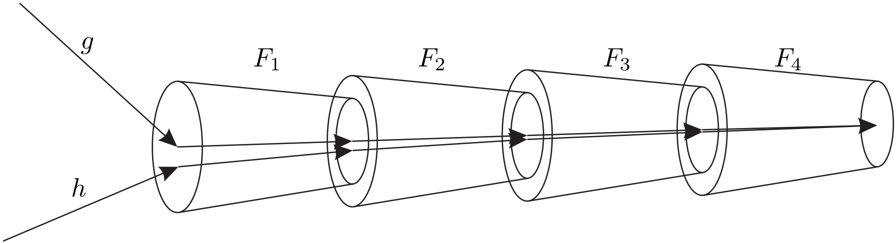




Let $\mathcal{D}(\mathfrak{G})$ be the set of maps $F : C \longrightarrow \overline{\mathfrak{G}_x^{\mathcal{X}'}}$ defined on a compact neighborhoods C of a point of $\partial\mathfrak{G}_x$ such that for any sequence $(g_n, h_n) \in C$ of different edges $d(h_n g_n^{-1}, F(h_n)F(g_n)^{-1}) \to 0$.

Let $\dots g_2 g_1 \cdot g$ be a point of $\partial\mathfrak{G}_x$, where $g_i \in S$. We can find contractions $F_i \in \widetilde{\mathfrak{G}}$ such that $g_i \in F_i$ and all compositions $F_n \cdots F_1$ are defined on a fixed neighborhood of range of g. If $h \in \mathfrak{G}$ is such that range of h is close enough to range of g, then we can define

$$F(\dots g_2 g_1 \cdot g) = \dots F_2 F_1 \cdot h$$



Duality Theorem

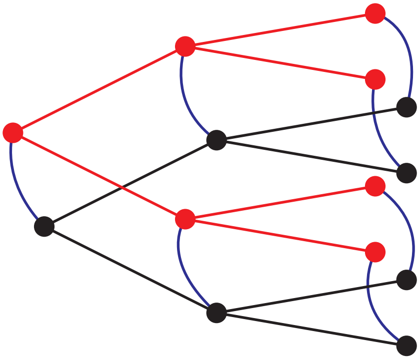
The set of germs of elements of $\mathcal{D}(\mathfrak{G})$ on $\partial\mathfrak{G}_x$ is the *dual groupoid* $\mathfrak{G}^\top$. It is uniquely defined, up to equivalence of groupoids.

Theorem

If $\mathfrak{G}$ is hyperbolic and minimal, then $\mathfrak{G}^\top$ is also hyperbolic and minimal, and $(\mathfrak{G}^\top)^\top$ is equivalent to $\mathfrak{G}$.

IMG as a dual groupoid

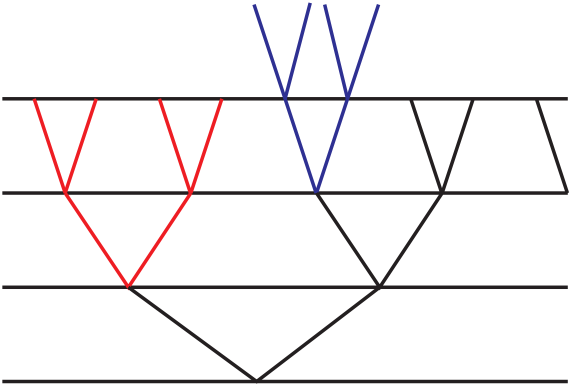
Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be an expanding self-covering. The Cayley graphs are trees of grand orbits. Paths in $\mathcal{X}$ define delements of $\mathcal{D}(\mathfrak{F})$.



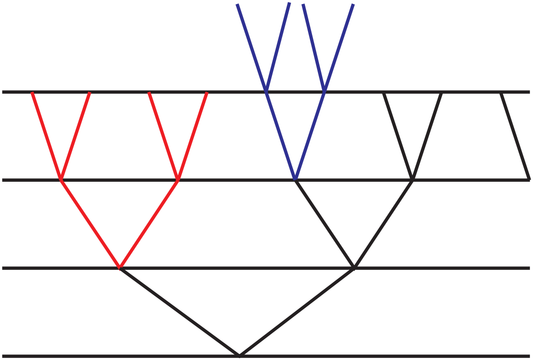
It follows that $\mathfrak{F}^\top$ is equivalent to the groupoid of germs generated by $\mathrm{IMG}(f)$ and the shift.

$(\mathcal{J}_G, \mathsf{s})$ as a dual groupoid

Let (G, X) be a regular contracting group. Let $\mathfrak{G}$ be the groupoid on X^ω generated by G and the shift. Consider the Cayley graph $\mathfrak{G}(w_0, S)$ for a fixed $w_0 \in \mathsf{X}^\omega$. For a given $w \in \mathfrak{G}(w_0, S)$ consider the *positive cone* P_w of points of the form $x_n x_2 \ldots x_1 w$.



Maps of the form $vw_1 \mapsto vw_2$ from P_{w_1} to P_{w_2} belong to $\mathcal{D}(\mathfrak{G})$, since the germ of g or T_x at vw_1 is close to the germ at vw_2, if v is long. The boundary $\partial\mathfrak{G}_{w_0}$ is locally homeomorphic to $\mathcal{J}_G$, and the germs of the maps $P_{w_1} \longrightarrow P_{w_2} : vw_1 \mapsto vw_2$ are identified with the germs of the groupoid generated by $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$.



Other examples

1. Actions of Gromov-hyperbolic groups on their boundaries are self-dual.

3. Let $\mathfrak{G}$ be the groupoid generated by a topological Markov chain defined by a matrix A. Then $\mathfrak{G}^\top$ is generated by a Markov chain defined by $A^\top$.

4. Groupoid generated by $x \mapsto x + 1$ and $x \mapsto 2x$ on $\mathbb{R}$ is dual to the groupoid generated by the same maps on $\mathbb{Z}_2$.

2. Ruelle groupoid acting on stable (contracted) leaves is dual to the Ruelle groupoid acting on the unstable (expanded) leaves of a Smale space.

Operator algebras

Theorem

Let $\mathfrak{G}$ be a minimal hyperbolic groupoid. Then $C^*(\mathfrak{G})$ is nuclear (?), simple, and purely infinite.

The theorem is a generalization of analogous results about Cuntz-Krieger algebras, algebras of boundary actions of hyperbolic groups, algebras generated by expanding maps, and Ruelle algebras.

K-theoretical Duality

The next conjecture would be a natural generalization of a result of J. Kaminker, I. Putnam, and M. Whittaker on Ruelle algebras of Smale spaces.

Conjecture

If $\mathfrak{G}$ is minimal hyperbolic, then $C^*(\mathfrak{G})$ and $C^*(\mathfrak{G}^\top)$ are Spanier-Whitehead dual of each other. That is, there exist duality classes $\delta \in KK^1(\mathbb{C}, C^*(\mathfrak{G}) \otimes C^*(\mathfrak{G}^\top))$ and $\Delta \in KK^1(C^*(\mathfrak{G}) \otimes C^*(\mathfrak{G}^\top), \mathbb{C})$ defining isomorphisms

$$K_*(C^*(\mathfrak{G})) \longrightarrow K^{*+1}(C^*(\mathfrak{G}^\top)), \qquad K^*(C^*(\mathfrak{G}^\top)) \longrightarrow K_{*+1}(C^*(\mathfrak{G})).$$

The conjecture is also known to be true for boundary actions of Gromov hyperbolic groups (H. Emerson).

The class $\Delta \in KK^1(C^*(\mathfrak{G}) \otimes C^*(\mathfrak{G}^\top), \mathbb{C})$ appearing in the works of J. Kaminker, I. Putnam, M. Whittaker, and H. Emerson is an extension of $C^*(\mathfrak{G}) \otimes C^*(\mathfrak{G}^\top)$ by compact operators, and comes directly from the definition of $\mathfrak{G}^\top$.