Definitions from topology

A function f is one-to-one if $f(x) \neq f(x)$ whenever $x \neq y$.

Let I and J be intervals and $f : I \rightarrow J$. The function f is *onto* if for any y in J there is an $x \in I$ such that $f(x) = y$.

A function $f : I \longrightarrow J$ is a *homeomorphism* if it is one-to-one, onto, continuous, and $f^{-1}(x)$ (the inverse transformation) is also continuous. (Continuity of the inverse will follow from the rest of the conditions if I and J are finite intervals.)

A homeomorphism $f : I \longrightarrow J$ is a C^r-diffeomorphism if f and f^{-1} has continuous derivatives of order r.

We denote

$$f^n(x) = \underbrace{f\,(f\,(f\,\ldots f}_{n \text{ times}}(x)\ldots)).$$

We also denote $f(g(x)) = f \circ g(x)$, so that

$$f^n(x) = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}(x).$$

Let $S \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is a *limit point* of S if there is a sequence of distinct points $x_n \in S$ converging to x. S is a closed set if it contains all of its limit points.

Let $S \subset \mathbb{R}$. S is an open set if, for any $x \in S$, there is an $\epsilon > 0$ such that all points t in the open interval $x - \epsilon < t < x + \epsilon$ are contained in S.

For any set A denote by $\overline{A}$ the closure of A, i.e., the set A together with the set of all limit points of A. It is the smallest closed set containing A.

Definition 1. A subset U of S is dense in S if $\overline{U} = S$.

Elementary Definitions

Let $f : I \longrightarrow I$ be a map. Dynamics studies iterations f^n of the map f. Here are some elementary vocabulary used in dynamics.

The *forward orbit* of x is the set $x, f(x), f^2(x), \ldots$.

A point x is a *fixed point* of f if $f(x) = x$.

A point x is a *periodic point* if $f^n(x) = x$ for some n. The smallest n is called the *period* of x.

A point x is *eventually periodic* if it is not periodic but there exists m such that $f^m(x)$ is periodic.

Let p be periodic of period n. A point x is *forward asymptotic to* p if $\lim_{k\to\infty} f^{nk}(x) = p$. The *stable set of* p is the set of all points forward asymptotic to p.

A point x is a *critical point* if $f'(x) = 0$. The critical point x is non-degenerate if $f''(x) \neq 0$.

Hyperbolicity

Let p be a periodic point of period n. The point p (or the corresponding cycle) is *hyperbolic* if $|(f^n)'(p)| \neq 1$. The number $(f^n)'(p)$ is called the *multiplier* of the cycle.

If $|(f^n)'(p)| < 1$, then the cycle is *attracting*. If $|(f^n)'(p)| > 1$, then it is repelling.

If the cycle is attracting, then all points sufficiently close to p are attracted to the cycle of p. If it is repelling, then all points close to p "run away" from the cycle.

More generally, a set $\Gamma \subset \mathbb{R}$ is a *repelling hyperbolic set* for f if Γ is closed, bounded and invariant under f and there exits $n > 0$ such that $|(f^n)'(x)| > 1$ for all $n \geq N$ and all $x \in \Gamma$.

If we change $|(f^n)'(x)| > 1$ to $|(f^n)'(x)| < 1$, then we get the definition of an *attracting* hyperbolic set.

Consider the function $f(x) = \mu x(1-x)$, where $\mu > 1$ is a parameter. This family of functions (depending on μ) is an interesting family of dynamical systems (on $\mathbb{R}$), which illustrates many aspects of more complex dynamical systems.

It has two fixed points 0 and $p_\mu = (\mu - 1)/\mu$. Zero is repelling and p_μ is attracting for $1 < \mu < 3$. As μ becomes bigger than 3, the function f^2 gets two more fixed points. The story becomes more complicated as μ changes from 3 to 4.

For all $\mu > 1$ the points outside of the interval $[0, 1]$ go to negative infinity under iterations of f, i.e., for all $x < 0$ and $x > 1$ we have $f^n(x) \to -\infty$ as $n \to \infty$.

If $1 < \mu < 3$, all points $x \in (0, 1)$ are attracted to p_μ.