M147, Fall 2009, Exam 3 Name______________________

Note for Fall 2010. This is Exam 3 for M147, Fall 2009, and the exam this year will have the same format (ten multiple choice questions and five problems for which work is to be written out). This exam covered two sections that will not be on Exam 3 for Fall 2010, Sections 6.1 and 6.2. In light of this, you should skip problems 7, 8, 9, 10, 14, and 15. On the other hand, you should go back and work problems 13, 14, and 15 from last year's Exam 2. Also, you will not need the summation formulas that were given on the exam last year. The WIR Tuesday November 23 will be exam review, and the relevant problems on this exam and last year's Exam 2 will be worked. Since this conflicts with Student Bonfire, solutions are also given at the end of this exam.

Calculators are not allowed on the exam. The first ten problems are multiple choice. Written out work on these problems will not be checked, so take care in marking your answers. For Problems 11-15 unjustified answers will not receive credit. You may need the following summation formulas.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \qquad \sum_{k=1}^{n} k^3 = \Big(\frac{n(n+1)}{2}\Big)^2.$$

1. [5 pts] Compute the limit

$$\lim_{x\to 0} \frac{1-e^{3x}}{x}.$$

(a) -3

(b) 3

(c) 0

(d) DNE

(e) None of the above

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2. [5 pts] Compute the limit

$$\lim_{x\to\infty} x\ln\Big(\frac{x}{x+1}\Big).$$

(a) 0

(b) 1

(c) -1

(d) DNE

(e) None of the above

3. [5 pts] Compute the limit

$$\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}.$$

(a) $\frac{1}{2}$

(b) $-\frac{1}{2}$

(c) $e^{\frac{1}{2}}$

(d) $e^{-\frac{1}{2}}$

(e) None of the above

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4. [5 pts] Compute the limit

$$\lim_{n \to \infty} (1 + \frac{2}{n})^n.$$

(a) e

(b) e^{-1}

(c) e^2

(d) e^{-2}

(e) None of the above

5. [5 pts] The recursion equation

$$x_{t+1} = x_t^2 - x_t - 3$$

has two fixed points $x^* = -1$ and $x^* = 3$. Classify each as unstable or asymptotically stable.

(a) Both are asymptotically stable

(b) Both are unstable

(c) $x^* = -1$ is asymptotically stable; $x^* = 3$ is unstable

(d) $x^* = 3$ is asymptotically stable; $x^* = -1$ is unstable

(e) None of the above (i.e., at least one test is indeterminate)

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6. [5 pts] Suppose x^* is a fixed point for the recursion equation

$$x_{t+1} = f(x_t),$$

with starting value x_0. If x^* is stable but not asymptotically stable then the following statement must be true:

(a) If x_0 is close to x^* then $\lim_{t\to\infty} x_t = x^*$.

(b) If x_0 is close to x^* then x_t remains close to x^* for all $t = 1, 2, 3, \ldots$

(c) If x_0 is close to x^* then $\lim_{t\to\infty} x_t = \infty$.

(d) If x_0 is *not* close to x^* then $\lim_{t\to\infty} x_t$ does not exist.

(e) None of the above

7. [5 pts] Evaluate the integral

$$\int_{-3}^{0}(4-\sqrt{9-x^2})dx.$$

(a) $12-\frac{9\pi}{4}$

(b) $12+\frac{9\pi}{4}$

(c) $6-\frac{9\pi}{4}$

(d) $6+\frac{9\pi}{4}$

(e) None of the above

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8. [5 pts] Compute the indefinite integral

$$\int \sin x(\cos^2 x+2)dx.$$

(a) $\sin^2 x(\cos^2 x+2)+C$

(b) $-\frac{1}{3}\cos^3 x-2\cos x+C$

(c) $-\frac{1}{3}\cos^3 x+2\cos x+C$

(d) $\frac{1}{3}\cos^3 x+2\cos x+C$

(e) None of the above

9. [5 pts] Evaluate the definite integral

$$\int_1^3 \frac{x}{\sqrt{9-x^2}}dx.$$

(a) $-2\sqrt{8}$

(b) $2\sqrt{8}$

(c) $-\sqrt{8}$

(d) $\sqrt{8}$

(e) None of the above

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10. [5 pts] Evaluate the definite integral

$$\int_{-3}^1 e^{3-|x|}dx.$$

(a) $2e^3 - e^2 - 1$

(b) $2e^3 + e^2 - 1$

(c) $2e^3 + e^2 + 1$

(d) $-2e^3 - e^2 - 1$

(e) None of the above

This problem continues on the next page.

11. [10 pts] Let

$$f(x) = \frac{x^2 - 2x + 4}{x - 2}, \quad x \neq 2.$$

11a. Locate the critical points of f and determine the intervals on which f is increasing and the intervals on which f is decreasing.

11b. Locate the possible inflection points for f and determine the intervals on which f is concave up and the intervals on which it is concave down.

11c. Evaluate f at the critical points, the possible inflection points, and identify its asymptotic behavior. Use this information to sketch a graph of this function.

12. [10 pts] A rectangular box with a top and square base is to be constructed at a cost of 20 cents. If the material for the bottom costs 3 cents per square foot, the material for the top costs 2 cents per square foot, and the material for the sides costs 1.5 cents per square foot, find the dimensions and volume of the box of maximum volume that can be constructed.

13. [10 pts] Find all fixed points for the recursion

$$x_{t+1} = 5 - \frac{1}{4}x_t,$$

and use the method of cobwebbing to find $\lim_{t\to\infty} x_t$ if $x_0 = 1$.

14. [5 pts each]

14a. Express the given integral as a Riemann sum. Be sure to define all quantities that appear in your expression.

$$\int_1^7 e^{\sqrt{1+x^2}}\,dx.$$

14b. Determine whether or not the fundamental theorem of calculus can be applied to the function

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{2} \\ 1 - \cos x, & \frac{\pi}{2} \le x \le \pi \end{cases},$$

on the interval $[0, \pi]$. Explain why or why not. (You need not integrate $f(x)$.)

15. [10 pts] Use the method of Riemann sums to evaluate

$$\int_2^4 x^2 dx.$$

Solutions

For 2009 Exam 2

13.

$$f'(x) = -\frac{1+x}{(x-1)^3},$$

and we conclude

$$\text{f is increasing on } [-1,1)$$
$$\text{f is decreasing on } (-\infty,-1] \cup (1,\infty).$$

14.

$$f''(x) = -\frac{4x}{(x^2-1)^2},$$

and we conclude

$$\text{f is concave up on } (-\infty,-1) \cup (-1,0)$$
$$\text{f is concave down on } (0,1) \cup (1,\infty).$$

15. This problem should additionally state that $f''(c) < 0$ at the maximum. (Recall that it's possible to have a maximum and have $f''(c) = 0$.) In this way, we know $f'(c) = 0$ and $f''(c) < 0$ (giving a local maximum by the second derivative test), and we need to show $g'(c) = 0$ and $g''(c) < 0$. We compute

$$g'(x) = \frac{f'(x)}{f(x)},$$

so that

$$g'(c) = \frac{f'(c)}{f(c)} = 0.$$

Likewise,

$$g''(x) = \frac{f''(x)f(x) - f'(x)^2}{f(x)^2},$$

so that

$$g''(c) = \frac{f''(c)}{f(c)} < 0.$$

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1. a

2. c

3. d

4. c

5. b

6. b

11. First,

$$f'(x) = \frac{x^2 - 4x}{(x-2)^2},$$

and we conclude

$$\begin{gathered}\text{f is increasing on } (-\infty, 0] \cup [4, \infty) \\ \text{f is decreasing on } [0, 2) \cup (2, 4].\end{gathered}$$

Next,

$$f''(x) = \frac{8}{(x-2)^3},$$

and we conclude

$$\begin{gathered}\text{f is concave down on } (-\infty, 2) \\ \text{f is concave up on } (2, \infty).\end{gathered}$$

The evaluations are, proceeding from left to right:

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= -\infty \\ f(0) &= -2 \\ \lim_{x\to 2^-} f(x) &= -\infty \\ \lim_{x\to 2^+} f(x) &= +\infty \\ f(4) &= 6 \\ \lim_{x\to+\infty} f(x) &= +\infty.\end{aligned}$$

The graph is given in Figure 1.

12. First, the cost equation is

$$5x^2 + 6xh = 20 \Rightarrow h = \frac{10}{3x} - \frac{5x}{6}.$$

The volume to maximize is

$$V = x^2 h = x^2\left(\frac{10}{3x} - \frac{5x}{6}\right) = \frac{10}{3}x - \frac{5}{6}x^3,$$

with $0 \le x \le 2$. (The upper limit puts all 20 cents worth of material into the base and top.) To find the critical points, we compute

$$V'(x) = \frac{10}{3} - \frac{5}{2}x^2 = 0 \Rightarrow x = \frac{2}{\sqrt{3}} ft.$$

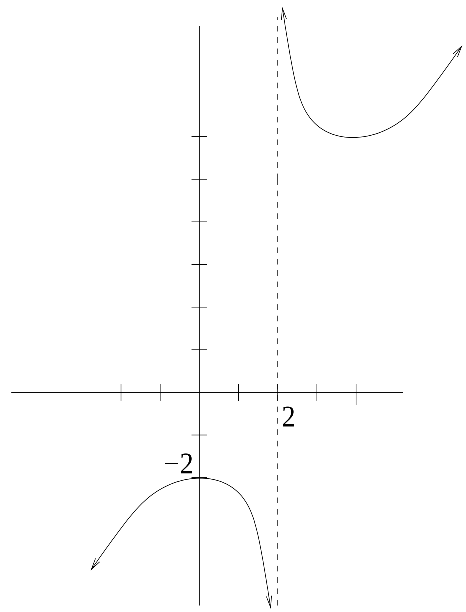

Figure 1: Figure for Problem 11.

To verify that this is the maximum, we compute

$$V(0) = 0$$
$$V(\frac{2}{\sqrt{3}}) = \frac{40}{9\sqrt{3}}$$
$$V(2) = 0.$$

Finally, to complete the dimensions, we observe that $h = \frac{10}{3\sqrt{3}}$.

13. The fixed points are solutions of

$$x^* = 5 - \frac{1}{4}x^* \Rightarrow x^* = 4.$$

The cobweb graph is depicted in Figure 2. We see that

$$\lim_{t\to\infty} x_t = 4.$$

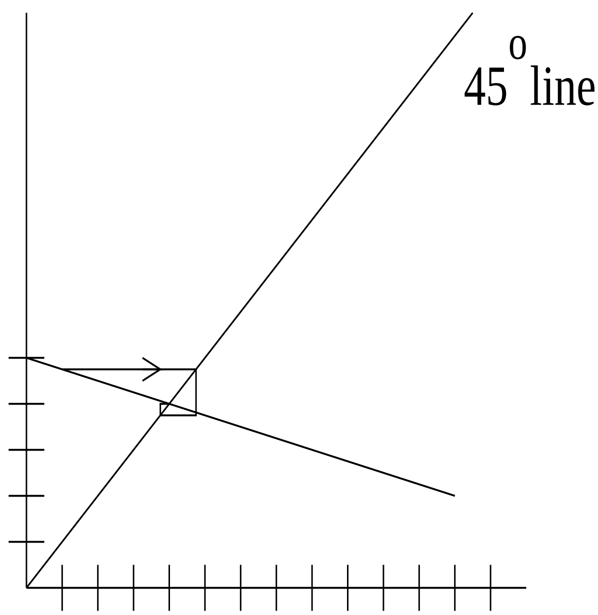

Figure 2: Figure for Problem 13