M147 Practice Problems for Exam 3

Exam 3 will cover sections 5.1, 5.2, 5.3, 5.4, 5.5, 2.1, 2.2, 2.3, and 5.6. Calculators will not be allowed on the exam. The first ten problems on the exam will be multiple choice. Work will not be checked on these problems, so you will need to take care in marking your solutions. For the remaining problems unjustified answers will not receive credit.

1. Sketch a graph of the function

$$f(x) = |3 - |x||,$$

on the interval $[-4, 1]$ and determine all local and global extrema on this interval.

2. For $f(x) = \frac{1}{x}$ on $[1, 5]$ the Mean Value Theorem asserts that there exists a value $c \in (1, 5)$ so that

$$f'(c) = -\frac{1}{5}.$$

Find c and depict this graphically.

3. Suppose that $f(x)$ is continuous on the interval $[2, 5]$ and differentiable on the interval $(2, 5)$. Show that if $1 \leq f'(x) \leq 4$ for all $x \in [2, 5]$, then $3 \leq f(5) - f(2) \leq 12$.

4a. For the function

$$f(x) = \frac{x^2}{1 + x}; \quad x \neq -1,$$

find the intervals on which f is increasing and the intervals on which f is decreasing.

4b. For the function defined in (a) find the intervals on which f is concave up and the intervals on which f is concave down.

5a. For the function

$$f(x) = x^{1/3}(1 - x)^{2/3},$$

find the intervals on which f is increasing and the intervals on which f is decreasing.

5b. For the function defined in (a) find the intervals on which f is concave up and the intervals on which f is concave down.

6. Suppose that $f(x)$ is twice differentiable in an open interval containing the point $x = c$ and has a local minimum at the same point, with $f''(c) > 0$. Show that the function $g(x) = e^{f(x)}$ has a local minimum at $x = c$.

7. Let

$$f(x) = \frac{e^{-x}}{x - 1}, \quad x \neq 1.$$

7a. Locate the critical points of f and determine the intervals on which f is increasing and the intervals on which f is decreasing.

7b. Locate the possible inflection points for f and determine the intervals on which f is concave up and the intervals on which f is concave down.

7c. Determine the boundary behavior of f by computing limits as $x \to \pm\infty$.

7d. Use your information from Parts a-c to sketch a graph of this function.

8. Let

$$f(x) = 3x^3 - 9x + 1, \quad x \in [-2, 2].$$

8a. Locate the critical points of f and determine the intervals on which f is increasing and the intervals on which f is decreasing.

8b. Locate the possible inflection points for f and determine the intervals on which f is concave up and the intervals on which f is concave down.

8c. Evaluate f at the critical points, possible inflection points, and boundary points.

8d. Use your information from Parts a-c to sketch a graph of this function.

9. Let

$$f(x) = 2x^{5/3} - 5x^{4/3}, \quad x \in \mathbb{R}.$$

9a. Locate the critical points of f and determine the intervals on which f is increasing and the intervals on which f is decreasing.

9b. Locate the possible inflection points for f and determine the intervals on which f is concave up and the intervals on which f is concave down.

9c. Evaluate f at the critical points, possible inflection points, and boundary points.

9d. Use your information from Parts a-c to sketch a graph of this function.

10. Find non-negative numbers x and y so that $x + y = 10$ and $y\sqrt{x}$ is maximized.

11. A long rectangular sheet of metal, 12 inches wide, is to be made into a rain gutter by turning up two sides at right angles to the sheet. How many inches should be turned up to give the gutter its greatest capacity?

12. A piece of wire 10 meters long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is minimized? How should the wire be cut so that the total area is maximized?

13. Compute the following limits.

13a.

$$\lim_{x \to 0} \frac{e^x - 1}{x}.$$

13b.

$$\lim_{x \to \infty} (x - \sqrt{x^2 - 1}).$$

13c.

$$\lim_{x \to 0} (\frac{1}{x} - \frac{1}{\tan x}).$$

13d.

$$\lim_{x \to \infty} (\frac{x}{x+1})^x.$$

13e.

$$\lim_{x\to 0^+} (\sin x)^x.$$

14. Write down a general expression a_n, $n = 0, 1, 2, \ldots$, for the sequence with terms

$$\frac{3}{2}, -\frac{5}{8}, \frac{7}{18}, -\frac{9}{32}, \frac{11}{50}, \ldots$$

15. Compute the limit

$$\lim_{n\to\infty} n^{\frac{1}{n}}.$$

16. Solve the recursion equation

$$a_{n+1} = \frac{1}{2}a_n - 1; \quad a_0 = -\frac{1}{2},$$

and compute $\lim_{x\to\infty} a_n$. Check your result by computing the fixed point for this equation.

17. Find all fixed points for the recursion

$$x_{t+1} = \frac{1}{2}x_t(\frac{1}{2} - x_t),$$

and use the method of cobwebbing to determine which limit will be achieved from the starting value $x_0 = -1$.

18. Find all fixed points for the recursion

$$x_{t+1} = \frac{4x_t^2}{\frac{7}{4} + x_t^2}.$$

On Figure 1, sketch cobweb graphs starting at $x_0 = 1$ and $x_0 = 5$. In each case, compute $\lim_{t\to\infty} x_t$.

19. Find all fixed points for the recursion

$$x_{t+1} = x_t e^{1-x_t},$$

and use the derivative condition for stability to determine whether each is stable or unstable.

20. The discrete logistic population model is

$$N_{t+1} = N_t + RN_t(1 - \frac{N_t}{K}).$$

Take $R = 1$ and $K = 10$ and show that one drawback of this model is that it can start with a positive population $N_0 > 0$ and return a negative population N_1.

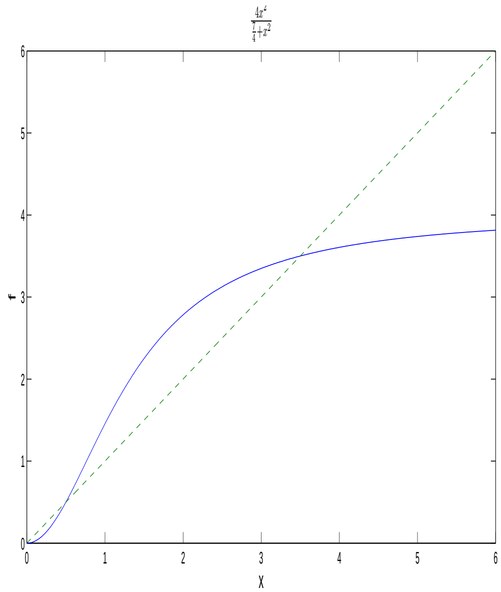

Figure 1: Graph for Problem 18.

Solutions.

1. First, the graph is given in Figure 2. One way to think about this is to recall

$$|x| = \begin{cases} -x & x \leq 0 \\ +x & x > 0. \end{cases}$$

and write

$$f(x) = |3 - |x|| = \begin{cases} |3 + x| & -4 \leq x \leq 0 \\ |3 - x| & 0 < x \leq 1 \end{cases} = \begin{cases} -(3 + x) & -4 \leq x \leq -3 \\ 3 + x & -3 < x \leq 0 \\ 3 - x & 0 < x \leq 1. \end{cases}$$

Each individual piece is easy to graph. Alternatively, you would be safe in this case simply plotting $f(x)$ for each integer $x = -4, -3, \ldots, 1$ and connecting the points with lines. The global minimum is $f(-3) = 0$ and the global maximum is $f(0) = 3$. The other local extrema are a local maximum at $f(-4) = 1$ and a local minimum $f(1) = 2$.

2. First,

$$f'(x) = -\frac{1}{x^2},$$

so

$$-\frac{1}{c^2} = -\frac{1}{5} \Rightarrow c = \sqrt{5}.$$

(Note that $c = \pm\sqrt{5}$, but we take the positive value because $c \in (1, 5)$.) The graph is given in Figure 3

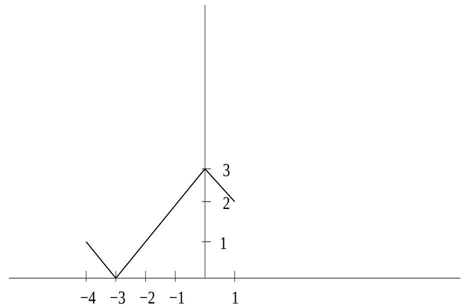

Figure 2: Figure for Problem 1 solution.

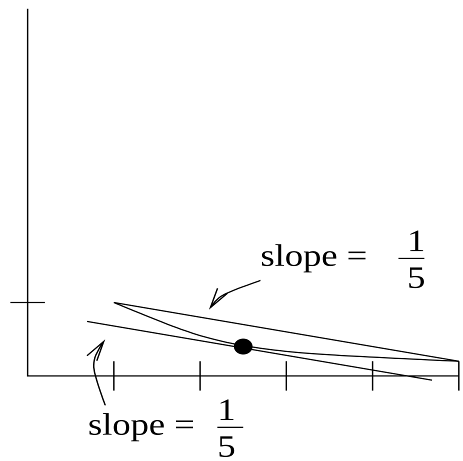

Figure 3: Figure for Problem 2.

3. By the Mean Value Theorem, we know that there exists some value $c \in (2, 5)$ so that

$$f'(c) = \frac{f(5) - f(2)}{3}.$$

Since the largest possible value for $f'(c)$ on this interval is 4 and since the smallest possible value for $f'(c)$ on this interval is 1, we have the inequality

$$1 \leq \frac{f(5) - f(2)}{3} \leq 4.$$

Multiplying this last inequality by 3, we find

$$3 \leq f(5) - f(2) \leq 12.$$

4a. First,

$$f'(x) = \frac{(1+x)2x - x^2}{(1+x)^2} = \frac{x^2 + 2x}{(1+x)^2},$$

and from this we can identify the critical points are $x = -2, -1, 0$. Plotting this on a number line, we find that f is decreasing on $[-2, -1) \cup (-1, 0]$ (note that the point where f

is undefined is excluded, but the other endpoints are included), and increasing on $(-\infty, -2] \cup [0, +\infty)$.

4b. We compute

$$f''(x) = \frac{(1+x)^2(2x+2) - (x^2+2x)2(1+x)}{(1+x)^4} = \frac{2(x+1)^2 - 2(x^2+2x)}{(1+x)^3} = \frac{2}{(1+x)^3}.$$

The only possible point of inflection is $x = -1$, and plotting this on a number line we find f is concave down on $(-\infty, -1)$ and concave up on $(-1, +\infty)$.

5a. Here,

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-2/3}(1-x)^{2/3} + x^{1/3}\frac{2}{3}(1-x)^{-1/3}(-1) \\ &= \frac{\frac{1}{3} - x}{x^{2/3}(1-x)^{1/3}}, \end{aligned}$$

and the critical points are $0, \frac{1}{3}$, and 1. We find

$$\begin{aligned} &\text{f is increasing on } (-\infty, \frac{1}{3}] \cup [1, \infty) \\ &\text{f is decreasing on } [\frac{1}{3}, 1]. \end{aligned}$$

5b. Working from (a), we compute

$$\begin{aligned} f''(x) &= \frac{-x^{2/3}(1-x)^{1/3} - (\frac{1}{3} - x)\Big(\frac{2}{3}x^{-1/3}(1-x)^{1/3} + \frac{1}{3}x^{2/3}(1-x)^{-2/3}(-1)\Big)}{x^{4/3}(1-x)^{2/3}} \\ &= -\frac{\frac{2}{9}}{x^{5/3}(1-x)^{4/3}}, \end{aligned}$$

where to get the second expression we multiplied the numerator and denominator of the first by $x^{1/3}(1-x)^{2/3}$. The possible inflection points are 0 and 1. We conclude

$$\begin{aligned} &\text{f is concave up on } (-\infty, 0) \\ &\text{f is concave down on } (0, \infty). \end{aligned}$$

6. By our assumptions on f, we know $f'(c) = 0$, and we are also given that $f''(c) < 0$. We need to show that precisely the same two conditions hold for $g(x) = e^{f(x)}$. We have

$$g'(x) = e^{f(x)} f'(x),$$

from which we see that

$$g'(c) = e^{f(c)} f'(c) = 0.$$

Next,

$$g''(x) = e^{f(x)} f'(x)^2 + e^{f(x)} f''(x),$$

from which we see that

$$g''(c) = e^{f(c)} f'(c)^2 + e^{f(c)} f''(c) = e^{f(c)} f''(c) < 0,$$

where we have used the observation that $f'(c) = 0$ and the fact that $e^{f(c)} > 0$.

7a. Compute

$$f'(x) = -\frac{xe^{-x}}{(x-1)^2},$$

and observe that the critical points are $x = 0, 1$. We see that f is increasing on $(-\infty, 0]$ and decreasing on $[0, 1) \cup (1, \infty)$. (We omit $x = 1$ because f isn't defined there.)

7b. Compute

$$f''(x) = \frac{e^{-x}(x^2+1)}{(x-1)^3},$$

and observe that the only possible inflection point is $x = 1$. We see that $f(x)$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

7c. For the boundary behavior, we have

$$\begin{aligned} \lim_{x\to-\infty} \frac{e^{-x}}{x-1} &= -\infty \\ \lim_{x\to+\infty} \frac{e^{-x}}{x-1} &= 0. \end{aligned}$$

7d. In order to anchor the plot, we evaluate f at the critical points, the possible inflection points and the endpoints. We have

$$\begin{aligned} f(0) &= -1 \\ \lim_{x\to1^-} \frac{e^{-x}}{x-1} &= -\infty \\ \lim_{x\to1^+} \frac{e^{-x}}{x-1} &= +\infty. \end{aligned}$$

The boundary behavior was obtained in Part c. The graph is sketched in Figure 4.

8a. First,

$$f'(x) = 9x^2 - 9,$$

so that the critical points are $x = \pm 1$. We find that

$$\begin{aligned} &\text{f is increasing on } [-2, -1] \cup [1, 2] \\ &\text{f is decreasing on } [-1, 1]. \end{aligned}$$

8b. Next,

$$f''(x) = 18x,$$

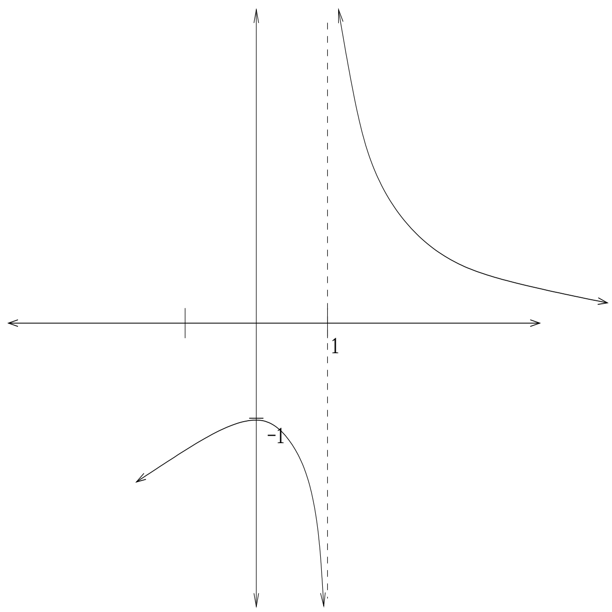

Figure 4: Figure for Problem 7.

so the only possible inflection point is $x = 0$. We find that

$$\begin{aligned} &\text{f is concave up on } (0, \infty) \\ &\text{f is concave down on } (-\infty, 0). \end{aligned}$$

8c. The evaluations are

$$\begin{aligned} f(-2) &= -5 \\ f(-1) &= 7 \\ f(0) &= 1 \\ f(1) &= -5 \\ f(2) &= 7. \end{aligned}$$

8d. The graph is in Figure 5.

9a. We compute

$$f'(x) = \frac{10}{3}x^{2/3} - \frac{20}{3}x^{1/3} = \frac{10}{3}x^{1/3}(x^{1/3} - 2),$$

so the critical points are $x = 0, 8$. We find

$$\begin{aligned} &\text{f is increasing on } (-\infty, 0] \cup [8, \infty) \\ &\text{f is decreasing on } [0, 8]. \end{aligned}$$

9b. Next,

$$f''(x) = \frac{20}{9}x^{-1/3} - \frac{20}{9}x^{-2/3} = \frac{20x^{1/3} - 20}{9x^{2/3}},$$

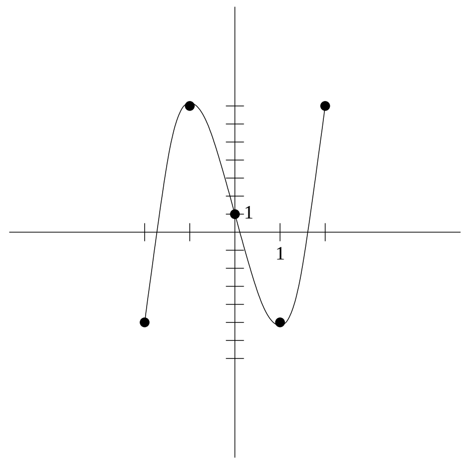

Figure 5: Figure for Problem 8.

and we see that the possible inflection points are $x = 0, 1$. We find

$$\text{f is concave up on } (1, \infty)$$
$$\text{f is concave down on } (-\infty, 0) \cup (0, 1).$$

9c. The evaluations are as follows:

$$\begin{aligned}
\lim_{x \to -\infty} f(x) &= -\infty \\
f(0) &= 0 \\
f(1) &= -3 \\
f(8) &= -16 \\
\lim_{x \to \infty} f(x) &= +\infty.
\end{aligned}$$

The limits are clear since for x large $x^{5/3} > x^{4/3}$.

9d. The graph is given in Figure 6.

10. If we write $y = 10 - x$, we want to maximize

$$f(x) = \sqrt{x}(10 - x) = 10\sqrt{x} - x^{3/2},$$

with $0 \leq x \leq 10$. We compute

$$f'(x) = \frac{10}{2\sqrt{x}} - \frac{3}{2}\sqrt{x} = \frac{10 - 3x}{2\sqrt{x}},$$

so that the critical points are $x = 0, \frac{10}{3}$. Finally, to see where the maximum occurs, we compute

$$\begin{aligned}
f(0) &= 0 \\
f(\frac{10}{3}) &= \sqrt{\frac{10}{3}}(10 - \frac{10}{3}) = \sqrt{\frac{10}{3}} \cdot \frac{20}{3} \\
f(10) &= 0.
\end{aligned}$$

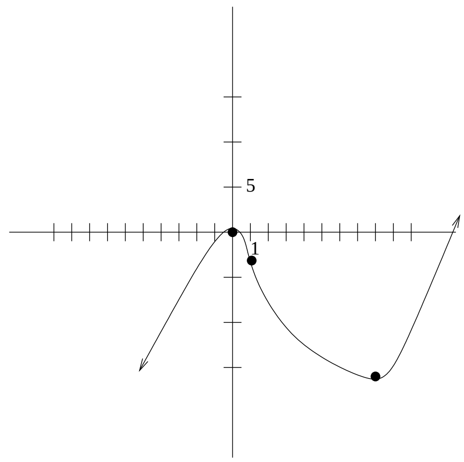

Figure 6: Graph for Problem 9.

We conclude that the numbers are $\frac{10}{3}$ and $\frac{20}{3}$.

11. Let x denote the width of sheet to be turned up on one side and let y denote the width of sheet left flat. If the length of the sheet is L then the volume is

$$V = xyL,$$

where y can be eliminated by the relation $y = 12 - 2x$. (For this problem it seems fairly natural to avoid bringing up the variable y, but I've used it here for consistency with our standard process. Also, it's clearly reasonable to omit L and work with the area $A(x) = xy$.) In this way the function we would like to maximize is

$$V(x) = x(12 - 2x)L, \quad 0 \leq x \leq 6.$$

We find the critical points by computing

$$\frac{dV}{dx} = (12 - 4x)L = 0 \Rightarrow x = 3.$$

Evaluating

$$\begin{aligned} V(0) &= 0 \\ V(3) &= 18L \\ V(6) &= 0, \end{aligned}$$

we conclude that the maximum capacity occurs when $x = 3$ inches are turned up on either side.

12. Let x be the length of each side of the square, and let y be the length of each side of the equilateral triangle. Then the total length of wire is

$$10 = 4x + 3y,$$

while the total area is

$$A = \text{area of square } + \text{ area of triangle} = x^2 + \frac{\sqrt{3}}{4}y^2.$$

(You can derive the area formula for an equilateral triangle from the formula $\frac{1}{2}bh$ and either the sidelengths for a 30-60-90 triangle or the Pythagorean theorem. See figure.)

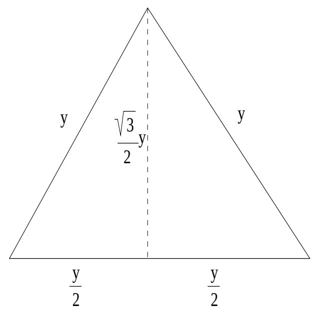

Figure 7: Figure for Problem 12.

Solving our constraint for y, we have

$$y = \frac{10}{3} - \frac{4}{3}x,$$

so that

$$A(x) = x^2 + \frac{\sqrt{3}}{4}(\frac{10}{3} - \frac{4}{3}x)^2, \quad 0 \le x \le \frac{10}{4}.$$

Proceeding as usual, we compute

$$A'(x) = 2x + \frac{\sqrt{3}}{2}(\frac{10}{3} - \frac{4}{3}x)(-\frac{4}{3}) \Rightarrow x(2 + \frac{8\sqrt{3}}{9}) = \frac{20\sqrt{3}}{9}.$$

We conclude that

$$x = \frac{20\sqrt{3}}{18 + 8\sqrt{3}} = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

From our expression for $A'(x)$ we see that $A'(x) < 0$ for $x < \frac{10\sqrt{3}}{9+4\sqrt{3}}$, while $A'(x) > 0$ for $x < \frac{10\sqrt{3}}{9+4\sqrt{3}}$. We conclude that $A(x)$ decreases for all x to the left of this point and increases for all x to the right of it, and is consequently a global minimum. Notice particularly that

$$0 < \frac{10\sqrt{3}}{9 + 4\sqrt{3}} < \frac{10\sqrt{3}}{4\sqrt{3}} = \frac{10}{4},$$

so this value of x is on our domain $0 \le x \le \frac{10}{4}$. This says that the area is minimized if the length of wire taken for the square is

$$4x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}}.$$

In order to find the global maximum, we must check $A(x)$ at the two endpoints. We have

$$\begin{aligned} A(0) &= \frac{100\sqrt{3}}{36} \\ A(\frac{10}{4}) &= \frac{100}{16}. \end{aligned}$$

Notice that

$$\frac{100\sqrt{3}}{36} < \frac{100 \cdot 2}{36} = \frac{100}{18} < \frac{100}{16},$$

so $A(\frac{10}{4})$ is larger, and this corresponds with putting all of the wire into the square.

13a.

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1.$$

13b.

$$\lim_{x\to\infty} (x - \sqrt{x^2 - 1}) = \lim_{x\to\infty} x(1 - \sqrt{1 - \frac{1}{x^2}}) = \lim_{x\to\infty} \frac{1 - \sqrt{1 - \frac{1}{x^2}}}{\frac{1}{x}}.$$

We can now apply l'Hospital's rule to find that this limit is

$$\lim_{x\to\infty} \frac{-\frac{1}{2}(1 - \frac{1}{x^2})^{-\frac{1}{2}}(\frac{2}{x^3})}{-\frac{1}{x^2}} = 0.$$

13c. In this case, we begin by writing $\tan x = \frac{\sin x}{\cos x}$ and finding a common denominator.

$$\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\tan x}) = \lim_{x\to 0}(\frac{1}{x} - \frac{\cos x}{\sin x}) = \lim_{x\to 0} \frac{\sin x - x\cos x}{x \sin x}.$$

We can now apply L'Hospital's rule repeatedly

$$\begin{aligned} \lim_{x\to 0} \frac{\cos x - \cos x + x\sin x}{\sin x + x\cos x} &= \lim_{x\to 0} \frac{\sin x + x\cos x}{\cos x + \cos x - x\sin x} \\ &= 0. \end{aligned}$$

13d.

$$\lim_{x\to\infty} (\frac{x}{x+1})^x = \lim_{x\to\infty} e^{\ln(\frac{x}{x+1})^x} = \lim_{x\to\infty} e^{x\ln(\frac{x}{x+1})} = e^{\lim_{x\to\infty} x\ln(\frac{x}{x+1})}.$$

We compute the limit in the exponent as

$$\lim_{x\to\infty} \frac{\ln(\frac{x}{x+1})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x(x+1)}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{x^2}{x^2 + x} = -1.$$

We conclude

$$\lim_{x\to\infty} (\frac{x}{x+1})^x = e^{-1}.$$

Note: It's slicker—but for our purposes less instructive—to simply notice that this is the inverse of the limit

$$e = \lim_{x\to\infty}(1+\frac{1}{x})^x.$$

13e. We have

$$\begin{aligned}\lim_{x\to 0^+}(\sin x)^x &= \lim_{x\to 0^+} e^{\ln(\sin x)^x} = \lim_{x\to 0^+} e^{x\ln(\sin x)} = e^{\lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{x}}}\\ &= e^{\lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}} = e^{\lim_{x\to 0^+} -\frac{x^2\cos x}{\sin x}} = e^0 = 1,\end{aligned}$$

where the final limit can be computed either using L'Hospital's rule once more or using our known trig limits.

14. First, we get the sign right with $(-1)^n$, $n = 0, 1, 2, \dots$, and we observe that the numerator is $2n+3$, for $n = 0, 1, 2, \dots$. The easiest way to understand the denominator is to factor out the common factor 2 (a useful trick in general). We find

$$a_n = (-1)^n \frac{2n+3}{2(n+1)^2}, \quad n = 0, 1, 2, \dots$$

15. The important thing to remember here is simply that you compute this sort of limit precisely as with limits in x. More precisely,

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{x\to\infty} x^{\frac{1}{x}}.$$

For the limit in x we can apply l'Hospital's Rule:

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln x^{\frac{1}{x}}} = e^{\lim_{x\to\infty}\frac{1}{x}\ln x} = e^{\lim_{x\to\infty}\frac{1}{x}} = e^0 = 1.$$

16. We solve this by iterating

$$\begin{aligned}a_1 &= \frac{1}{2}(-\frac{1}{2}) - 1 = -\frac{5}{4}\\ a_2 &= \frac{1}{2}(-\frac{5}{4}) - 1 = -\frac{13}{8}\\ a_3 &= \frac{1}{2}(-\frac{13}{8}) - 1 = -\frac{29}{16}\\ &\vdots\end{aligned}$$

We recognize the pattern as

$$a_n = -\frac{2^{n+2}-3}{2^{n+1}}.$$

For the limit, we compute

$$\lim_{n\to\infty} a_n = -2,$$

which is also the fixed point for this equation.

17. First, the fixed point equation is

$$x = \frac{1}{2}x(\frac{1}{2} - x) = \frac{1}{4}x - \frac{1}{2}x^2,$$

so that the fixed points are

$$\frac{3}{4}x + \frac{1}{2}x^2 = x(\frac{3}{4} + \frac{1}{2}x) = 0 \Rightarrow x = 0, -\frac{3}{2}.$$

For the cobwebbing, we can plot $f(x) = \frac{1}{4}x - \frac{1}{2}x^2 = x(\frac{1}{4} - \frac{1}{2}x)$ by noticing that it's a parabola opening downward with x-intercepts at $x = 0$ and $x = \frac{1}{2}$, and therefore has a maximum value at (the midpoint) $\frac{1}{4}$ of $\frac{1}{4}(\frac{1}{4} - \frac{1}{2}\frac{1}{4}) = \frac{1}{4}\frac{1}{8} = \frac{1}{32}$. We find that for $x_0 = -1$ (see the figure)

$$\lim_{t\to\infty} x_t = 0.$$

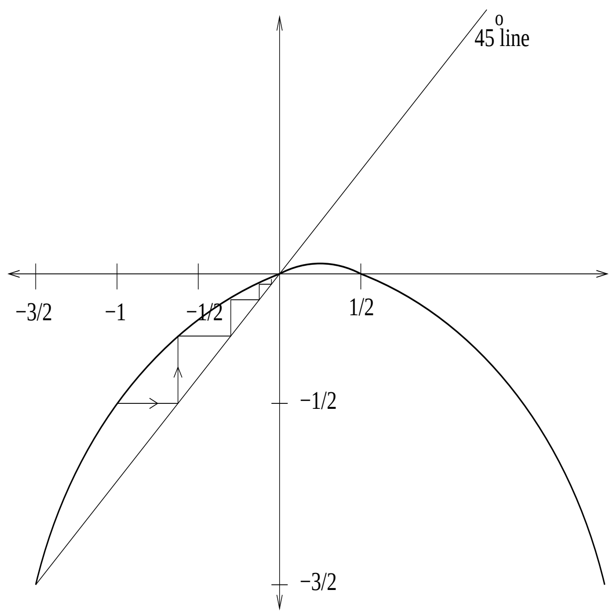

Figure 8: Figure for Problem 17.

18. First, the fixed points are solutions of

$$x = \frac{4x^2}{\frac{7}{4} + x^2}.$$

First, we recognize that $x = 0$ works, and then divide by x to get

$$1 = \frac{4x}{\frac{7}{4} + x^2} \Rightarrow x^2 + \frac{7}{4} = 4x.$$

In this way, the fixed points are solutions of the quadradic equation

$$x^2 - 4x + \frac{7}{4} = 0.$$

The roots are

$$x = \frac{4 \pm \sqrt{4^2 - 4 \cdot \frac{7}{4}}}{2} = 2 \pm \frac{\sqrt{9}}{2} = \frac{1}{2}, \frac{7}{2}.$$

(Notice that these values agree with the given figure.) The cobweb graph is sketched in Figure 9. We see that in both cases of x_0

$$\lim_{t \to \infty} x_t = \frac{7}{2}.$$

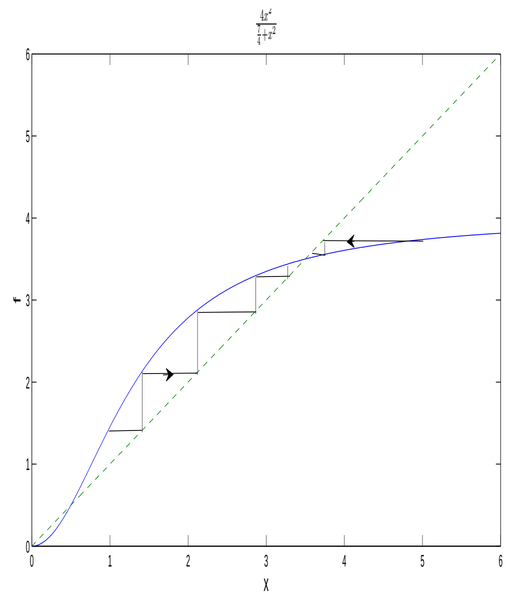

Figure 9: Figure for Problem 18 solution.

19. First, in order to find the fixed points we solve

$$x = xe^{1-x} \Rightarrow x(1 - e^{1-x}) = 0,$$

from which we have the fixed points

$$x = 0, 1.$$

In order to evaluate the stability of these points, we set

$$f(x) = xe^{1-x},$$

and compute

$$f'(x) = e^{1-x} + xe^{1-x}(-1) = e^{1-x}(1 - x).$$

We have:

$$f'(0) = e \Rightarrow |f'(0)| > 1 \Rightarrow x = 0 \text{ is unstable}$$
$$f'(1) = 0 \Rightarrow |f'(0)| < 1 \Rightarrow x = 1 \text{ is asymptotically stable}$$

20. First, for $R = 1$ and $K = 10$ the model becomes

$$N_{t+1} = N_t + N_t(1 - \frac{N_t}{10}).$$

We see that if N_t is large the second term will be negative, and as a convenient value we can take $N_0 = 50$. We find

$$N_1 = 50 + 50(1 - 5) = 50 - 200 = -150.$$