# M147 Practice Problems for Final Exam

The final exam for M147 will be on Wednesday, Dec. 15, 1:00-3:00 p.m., in your usual lecture room. Calculators will not be allowed on the exam. The first ten problems on the exam will be multiple choice. Work will not be checked on these problems, so you will need to take care in marking your solutions. For the remaining problems unjustified answers will not receive credit. On the exam you will be given the following identities:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \qquad \sum_{k=1}^{n} k^3 = \Big(\frac{n(n+1)}{2}\Big)^2.$$

**The exam will cover the following topics:**

1. Semilog and double-log plots (Section 3.1)
2. Limits of functions (Sections 3.1, 3.3, 3.4)
3. Continuity of functions (Section 3.2; The bisection method won't be on the exam)
4. Differentiation (All sections in Chapter 4)
5. Applications of differentiation (Sections 5.1, 5.2, 5.3, 5.4, 5.5)
6. Sequences, recursions, fixed points, and stability (Sections 2.1, 2.2, 2.3, 5.6)
7. Integration (Sections 6.1, 6.2, 7.1)

**You should memorize the following derivative formulas:**

1. $\frac{d}{dx} x^r = r x^{r-1}$ for any real number $r$
2. $\frac{d}{dx} \sin x = \cos x$
3. $\frac{d}{dx} \cos x = -\sin x$
4. $\frac{d}{dx} \tan x = \sec^2 x$
5. $\frac{d}{dx} e^x = e^x$
6. $\frac{d}{dx} a^x = a^x \ln a$, $a > 0$ (which contains (5) as the case $a = e$)
7. $\frac{d}{dx} \ln x = \frac{1}{x}$
8. $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$, $a > 0$, $a \neq 0$ (which contains (7) as the case $a = e$)
9. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
10. $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
11. $\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2+1}$

**Also, be sure you are able to use the following rules of differentiation:**

1. Product rule: $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$
2. Quotient rule: $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
3. Chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

4. Derivative of a function inverse: $\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}$.

**You should memorize the following integration formulas:**

1. $\int x^r dx = \frac{x^{r+1}}{r+1} + C$, $r \neq -1$
2. $\int \frac{1}{x} dx = \ln|x| + C$
3. $\int e^x dx = e^x + C$
4. $\int a^x dx = \frac{a^x}{\ln a} + C$, $a > 0$, $a \neq 1$
5. $\int \sin x dx = -\cos x + C$
6. $\int \cos x dx = \sin x + C$
7. $\int \sec^2 x dx = \tan x + C$
8. $\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$
9. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

1. Use a logarithmic transformation to find a linear relationship between (appropriate transformations of) $x$ and $y$ if

$$y = 3x^{-7}.$$

2. Given the semilog plot in Figure 1, find a functional relationship between $x$ and $y$.

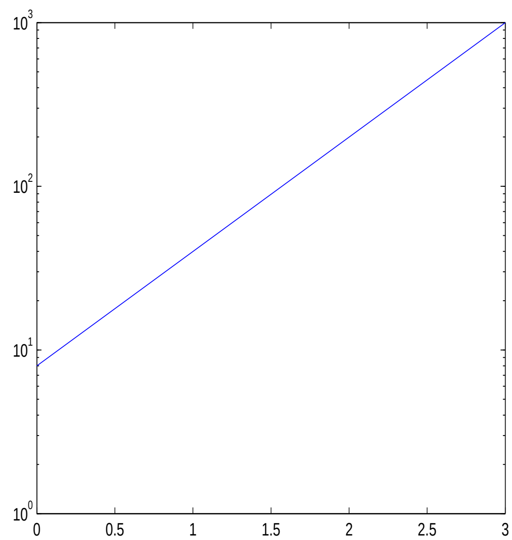


Figure 1: Figure for Problem 2.

3. Given the double-log plot in Figure 2, find a functional relationship between $x$ and $y$.

4. Compute each of the following limits:

4a.

$$\lim_{x \to 2^-} \frac{x}{x^2 + 3x - 10}.$$

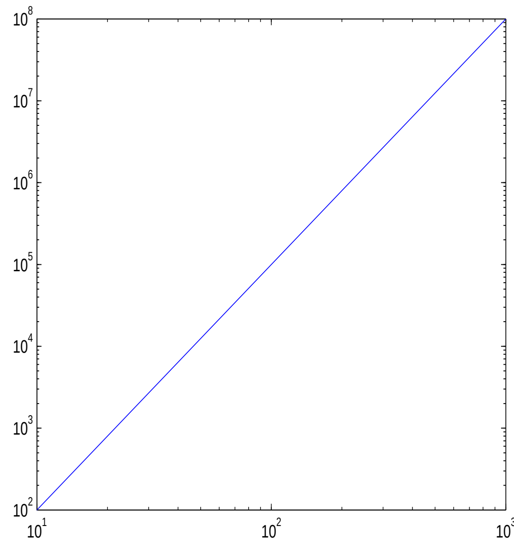


Figure 2: Figure for Problem 3.

4b.

$$\lim_{x\to 0} x e^{\sin(\frac{1}{x})}.$$

4c.

$$\lim_{x\to 0} \frac{x \sin x}{(1-e^x)^2}.$$

4d.

$$\lim_{x\to\infty} [(x+1)^{1/3} - x^{1/3}].$$

4e. The *geometric mean* of two positive real numbers $a$ and $b$ is defined as $\sqrt{ab}$. Show that

$$\sqrt{ab} = \lim_{x\to\infty} (\frac{a^{1/x} + b^{1/x}}{2})^x.$$

5a. Find a value for $c$ that makes the given function continuous at all points.

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}.$$

5b. Determine whether or not your function from (a) is differentiable at $x = 0$. If it is differentiable at this point, compute its derivative there.

6. Compute the derivative of each of the following functions.

6a.

$$f(x) = \frac{1 + \sin^2 x}{x \cos x}.$$

6b.

$$f(x) = \tan^{-1}(2^{\sqrt{x^2+1}}).$$

7. Find an equation for the line that is tangent to the graph of

$$f(x) = \frac{xe^x}{1+x^2}$$

at the point $x = 0$.

8. According to the Michaelis-Menton equation, when a chemical reaction involving a substrate $S$ is catalyzed by an enzyme, the rate of reaction is

$$R = \frac{as}{b+s},$$

where $s$ denotes substrate concentration (for example, in moles per liter), and $a$ and $b$ are constants. For this problem we'll take $a = b = 1$, for which we have

$$R = \frac{s}{1+s}.$$

If substrate is added at a rate $\frac{ds}{dt} = .1$ mol/L-s, find the rate at which $R$ is changing when $s = 9$ mol/L.

9. Suppose $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval $(a, b)$. Show that if $f'(x) = x$ for all $x \in (a, b)$, then there exists some value $c \in (0, 1)$ so that

$$f(b) - f(a) = c(b - a).$$

10. Let

$$f(x) = x^{1/3}(x+3)^{2/3}, \quad -\infty < x < \infty.$$

10a. Locate the critical points of $f$ and determine the intervals on which $f$ is increasing and the intervals on which $f$ is decreasing.

10b. Locate the possible inflection points for $f$ and determine the intervals on which $f$ is concave up and the intervals on which $f$ is concave down.

10c. Evaluate $f$ at the critical points and at the possible inflection points, and determine the boundary behavior of $f$ by computing limits as $x \to \pm\infty$.

10d. Use your information from Parts a-c to sketch a graph of this function.

11. Find the largest possible area of a right triangle whose hypotenuse is 2 cm.

12. Find the side-lengths that maximize the area of an isosceles triangle with given perimeter $P = 10$. (An isosceles triangle is a triangle with two sidelengths equal.)

13. Find all fixed points for the recursion

$$x_{t+1} = \frac{2}{1+x_t}.$$

Sketch a cobweb graph starting at $x_0 = 0$, and use it to determine $\lim_{t\to\infty} x_t$ in this case. (Your graph can be restricted to $x > -1$.)

14. Find all fixed points for the recursion

$$x_{t+1} = x_t e^{\frac{1}{2}(1-x_t^3)}.$$

Sketch cobweb graphs starting at $x_0 = \frac{1}{2}$ and $x_0 = \frac{3}{2}$ on Figure 3, and use these graphs to evaluate $\lim_{t\to\infty} x_t$ in each case. (In Figure 3, $f(x) = xe^{\frac{1}{2}(1-x^3)}$ is plotted along with a 45 degree line.)

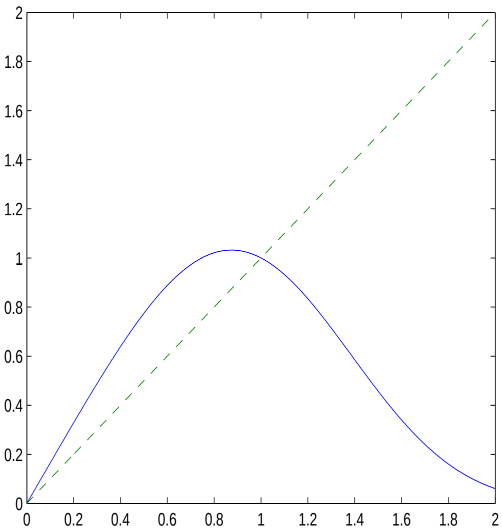


Figure 3: Figure for Problem 14.

15. Find all fixed points for the recursion equation

$$x_{t+1} = 1 + \frac{2}{x_t}$$

and classify each as stable or unstable.

16. Use a geometric argument to evaluate the integral

$$\int_{-1}^{1} |x| dx.$$

17. Suppose a function $f(x)$ is continuous on the interval $[0, 1]$ and that you are given the values in Table 1. Use these values to approximate the value of $\int_0^1 f(x)dx$.

18. Use the method of Riemann sums to evaluate

$$\int_1^2 x + x^2 dx.$$

| $x$ | $f(x)$ |
|---|---|
| $1/8$ | $1/2$ |
| $3/8$ | $1/3$ |
| $5/8$ | $-1$ |
| $7/8$ | $-2$ |

Table 1: Values of $f(x)$ for Problem 17.

19. Use the method of Riemann sums to evaluate

$$\int_1^4 1 - x^2 dx.$$

20. Determine whether or not the Fundamental Theorem of Calculus can be applied to the function

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 < x \leq 2 \end{cases},$$

on the interval $[0, 2]$. If so, find the anti-derivative $F(x)$ and use it to compute

$$\int_0^2 f(x)dx.$$

21. Compute

$$\frac{d}{dx} \int_{\cos x}^{x^2+1} e^{-x^2} dx.$$

22. Evaluate the following indefinite integrals.

22a.

$$\int e^x \cos(e^x) dx.$$

22b.

$$\int \frac{x}{\sqrt{1+x}} dx$$

22c.

$$\int \cos(2x - 1) dx.$$

22d.

$$\int \frac{x}{x^2 + 1} dx.$$

23. Evaluate the following definite integrals.

23a.

$$\int_0^{\sqrt{\pi}} x \sin(x^2) dx.$$

23b.

$$\int_1^e \frac{\sqrt{\ln x}}{x} dx.$$

23c.

$$\int_{-\frac{\pi}{2}}^{\pi} \sin |x| dx.$$

23d.

$$\int_1^3 \frac{x^2}{\sqrt{1+x^3}} dx.$$

24. Evaluate the following indefinite integral

$$\int \frac{\sin^3 x \cos x}{\sqrt{1+\sin^2 x}} dx.$$

# Solutions

1. Though we can use a logarithm with any valid base (i.e, $a > 0$, $a \neq 1$), we usually use base 10 for problems like this, denoted simply log. Taking a base 10 logarithm of both sides, we find

$$\log y = \log 3x^{-7} = \log 3 + \log x^{-7} = \log 3 - 7 \log x.$$

We see that this is a linear relationship between $\log y$ and $\log x$ with slope -7 and y-intercept (here, technically $\log y$-intercept) $\log 3$.

2. Since this is a semilog plot, we look for a relationship of the form

$$\log y = mx + b.$$

Here,

$$m = \frac{\log 10^3 - \log 8}{3} = \frac{\log 10^3 - \log 2^3}{3} = \frac{\log \frac{10^3}{2^3}}{3} = \frac{\log 5^3}{3} = \log 5.$$

(You can solve the problem without recognizing this simplification.) In this case, we can read $b$ directly from the graph as $b = \log 8$. This means we have

$$\log y = x \log 5 + \log 8 = \log 5^x + \log 8.$$

Finally, we take each side as an exponent of 10:

$$10^{\log y} = 10^{\log 5^x + \log 8} = 10^{\log 5^x} 10^{\log 8} = 5^x \cdot 8.$$

We conclude

$$y = 8 \cdot 5^x.$$

3. Since this is a double-log plot, we look for a relationship of the form

$$\log y = m \log x + b,$$

where from the graph
$$m = \frac{\log 10^8 - \log 10^2}{\log 10^3 - \log 10^1} = \frac{8-2}{3-1} = 3.$$
In this case, we can't read the $y$-intercept directly from the graph (since we don't have the $x$ value $10^0 = 1$), so we use the point-slope form. In this case, I've chosen the point on the lower left corner, $(10^1, 10^2)$. The point $(10^3, 10^8)$ on the upper right corner would work just as well. We have
$$\log y - \log 10^2 = 3(\log x - \log 10^1) = 3\log x - 3 = \log x^3 - 3,$$
and so
$$\log y = \log x^3 - 3 + 2 = \log x^3 - 1.$$
Taking each side as an exponent of the base 10, we conclude
$$10^{\log y} = 10^{\log x^3 - 1} = 10^{\log x^3} 10^{-1} = \frac{1}{10}x^3,$$
or
$$y = \frac{1}{10}x^3.$$
4a. We have
$$\lim_{x\to 2^-} \frac{x}{x^2+3x-10} = \lim_{x\to 2^-} \frac{x}{(x-2)(x+5)} = -\infty,$$
where we have observed that $x-2$ is negative for $x$ to the left of 2.

4b. We apply the Squeeze Theorem in this case, using the inequality
$$-|x|e \le xe^{\sin(\frac{1}{x})} \le |x|e.$$
We have
$$\lim_{x\to 0} -|x|e = \lim_{x\to 0} |x|e = 0,$$
and so according to the Squeeze Theorem
$$\lim_{x\to 0} xe^{\sin(\frac{1}{x})} = 0.$$
4c. We apply L'Hospital's Rule twice,
$$\lim_{x\to 0}\frac{x\sin x}{(1-e^x)^2} = \lim_{x\to 0}\frac{\sin x + x\cos x}{2(1-e^x)(-e^x)} = \lim_{x\to 0}\frac{\sin x + x\cos x}{2e^{2x}-2e^x}$$
$$= \lim_{x\to 0}\frac{2\cos x - x\sin x}{4e^{2x}-2e^x} = 1.$$
4d. This limit has the indeterminate form $\infty - \infty$, so the first thing we do is rearrange it into an expression with the form $\frac{0}{0}$. We have
$$\lim_{x\to\infty}(x+1)^{1/3} - x^{1/3} = \lim_{x\to\infty} x^{1/3}[(1+\frac{1}{x})^{1/3} - 1]$$
$$= \lim_{x\to\infty}\frac{(1+\frac{1}{x})^{1/3}-1}{x^{-1/3}} = \lim_{x\to\infty}\frac{\frac{1}{3}(1+\frac{1}{x})^{-2/3}(-\frac{1}{x^2})}{-\frac{1}{3}x^{-4/3}}$$
$$= \lim_{x\to\infty}(1+\frac{1}{x})^{-2/3}\frac{1}{x^{2/3}} = 0.$$

4e. We observe that this limit has the general form $1^\infty$, and so we can apply L'Hospital's rule. We have

$$\lim_{x\to\infty}(\frac{a^{1/x}+b^{1/x}}{2})^x = \lim_{x\to\infty} e^{\ln(\frac{a^{1/x}+b^{1/x}}{2})^x} = \lim_{x\to\infty} e^{x\ln(\frac{a^{1/x}+b^{1/x}}{2})}$$
$$= e^{\lim_{x\to\infty} x\ln(\frac{a^{1/x}+b^{1/x}}{2})}.$$

In order to compute this limit, we write

$$\lim_{x\to\infty} x\ln(\frac{a^{1/x}+b^{1/x}}{2}) = \lim_{x\to\infty}\frac{\ln(\frac{a^{1/x}+b^{1/x}}{2})}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{2}{a^{1/x}+b^{1/x}}(\frac{1}{2}a^{1/x}(\ln a)(-\frac{1}{x^2})+\frac{1}{2}b^{1/x}(\ln b)(-\frac{1}{x^2}))}{-\frac{1}{x^2}}$$
$$= \lim_{x\to\infty}\frac{1}{a^{1/x}+b^{1/x}}(a^{1/x}\ln a + b^{1/x}\ln b) = \frac{1}{2}(\ln a+\ln b),$$

where in this last step we have used that $\frac{1}{x}\to 0$ as $x\to\infty$. The limit is

$$e^{\frac{1}{2}(\ln a+\ln b)} = e^{\frac{1}{2}\ln(ab)} = e^{\ln(ab)^{1/2}} = \sqrt{ab}.$$

5a. Since

$$\lim_{x\to 0}\frac{\sin x}{x} = 1,$$

we can make this function continuous at all points by choosing $c = 1$.

5b. Since the function is separately defined at $x = 0$, we must proceed from the definition of differentiation. We compute

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\frac{\sin h}{h}-1}{h}$$
$$= \lim_{h\to 0}\frac{\sin h - h}{h^2} = \lim_{h\to 0}\frac{\cos h - 1}{2h} = \lim_{h\to 0}\frac{-\sin h}{2} = 0,$$

where the last two steps both used L'Hospital's rule. We conclude that this function is differentiable at $x = 0$, and that $f'(0) = 0$.

6a. We combine the quotient rule with the product rule to compute

$$f'(x) = \frac{(2\sin x\cos x)x\cos x - (1+\sin^2 x)(\cos x - x\sin x)}{(x\cos x)^2}$$
$$= \frac{2x\sin x\cos^2 x - \cos x + x\sin x - \sin^2 x\cos x + x\sin^3 x}{(x\cos x)^2}.$$

At this point, we could combine terms, but the expression doesn't simply much.

6b. This is a nested chain rule. We have

$$f'(x) = \frac{1}{1+2^{2\sqrt{x^2+1}}}2^{\sqrt{x^2+1}}\frac{x\ln 2}{\sqrt{x^2+1}}.$$

Notice that we can simplify $2^{2\sqrt{x^2+1}}$ as $4^{\sqrt{x^2+1}}$.

7. First,

$$f'(x) = \frac{(e^x + xe^x)(1+x^2) - xe^x(2x)}{(1+x^2)^2} \Rightarrow f'(0) = 1,$$

which is the slope of the tangent line. Using $f(0) = 0$ and the general point-slope form $y - f(a) = f'(a)(x-a)$, we conclude

$$y = x.$$

8. We differentiate implicitly:

$$\begin{aligned} \frac{dR}{dt} &= \frac{\frac{ds}{dt}(1+s) - s\frac{ds}{dt}}{(1+s)^2} = \frac{\frac{ds}{dt}}{(1+s)^2} \\ &= \frac{.1}{100} = .001 \text{ mol/L-s}^2. \end{aligned}$$

9. According to the Mean Value Theorem there exists some value $c \in (a,b)$ so that

$$\frac{f(b) - f(a)}{b-a} = f'(c).$$

In this case $f'(c) = c$, and so we conclude

$$\frac{f(b) - f(a)}{b-a} = c \Rightarrow f(b) - f(a) = c(b-a).$$

10a. The derivative of $f(x)$ is

$$f'(x) = \frac{x+1}{x^{2/3}(x+3)^{1/3}},$$

from which we find the critical points $x = -3, -1, 0$. We see that $f$ is increasing on $(-\infty, -3] \cup [-1, \infty)$ and decreasing on $[-3, -1]$.

10b. The second derivative of $f(x)$ is

$$f''(x) = -\frac{2}{x^{5/3}(x+3)^{4/3}},$$

from which we find that the possible inflection points are $x = 0, -3$. We see that $f$ is concave up on $(-\infty, -3) \cup (-3, 0)$ and concave down on $(0, \infty)$.

10c. Evaluating $f$ at the critical points, possible inflection points, and at the endpoints, we have:

$$\begin{aligned} f(-3) &= 0 \\ f(-1) &= -2^{2/3} \\ f(0) &= 0 \\ \lim_{x \to -\infty} x^{1/3}(x+3)^{2/3} &= -\infty \\ \lim_{x \to \infty} x^{1/3}(x+3)^{2/3} &= +\infty. \end{aligned}$$

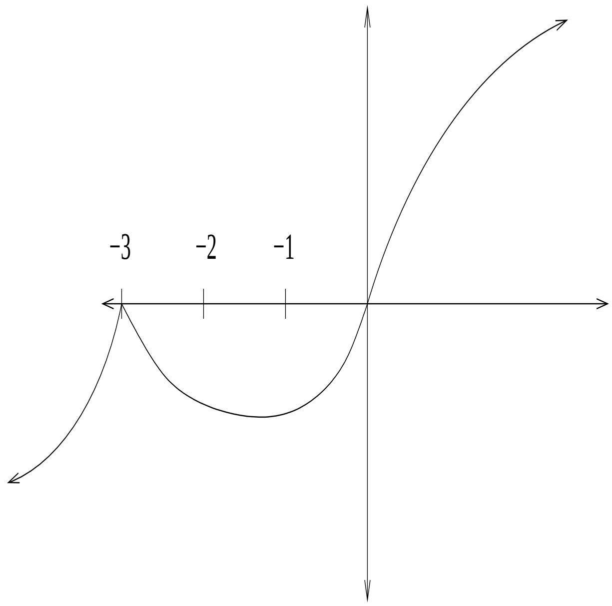


Figure 4: Figure for Problem 10 solution.

10d. Your plot should look something like Figure 4.

11. Let $x$ and $y$ denote the sidelengths, as indicated in Figure 5. The area of this triangle is

$$A = \frac{1}{2}xy,$$

and $x$ and $y$ are related by the Pythagorean Theorem,

$$x^2 + y^2 = 4 \Rightarrow y = \sqrt{4 - x^2}.$$

Here, we take the positive square root since $y$ is a length. We can now write $A$ entirely as a function of $x$,

$$A(x) = \frac{1}{2}x\sqrt{4 - x^2}, \quad 0 \le x \le 2.$$

We find the critical points by computing

$$A'(x) = \frac{1}{2}\sqrt{4 - x^2} + \frac{1}{2}x\frac{1}{2\sqrt{4 - x^2}}(-2x) = \frac{\frac{1}{2}(4 - x^2) - \frac{1}{2}x^2}{\sqrt{4 - x^2}} = \frac{2 - x^2}{\sqrt{4 - x^2}}.$$

The critical points are $x = \pm 2$, $\pm\sqrt{2}$. We discard the negative values because they're not on our interval, and we observe that $x = 2$ is already a right endpoint. Finally, the evaluations are

$$\begin{aligned} A(0) &= 0 \\ A(\sqrt{2}) &= 1 \\ A(2) &= 0. \end{aligned}$$

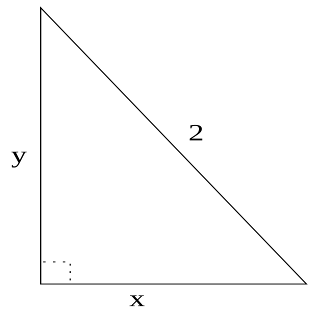


Figure 5: Figure for Problem 11 solution.

Clearly, the maximum occurs when $x = \sqrt{2}$, which corresponds with $y = \sqrt{2}$.

12. Let $y$ denote the length of the sides of equal length, and let $x$ denote the length of the side between them. (See Figure 6.) Then the perimeter is

$$10 = 2y + x \Rightarrow y = 5 - \frac{1}{2}x.$$

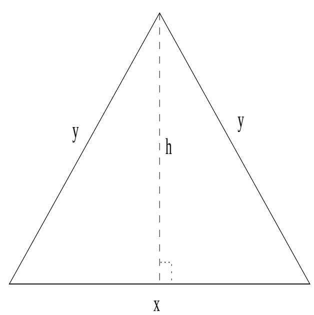


Figure 6: Figure for Problem 12 solution.

By the Pythagorean Theorem, $h^2 + (\frac{1}{2}x)^2 = y^2$, so the height of such a triangle is $h = \sqrt{y^2 - \frac{1}{4}x^2}$. The area to be maximized is

$$A = \frac{1}{2}x\sqrt{y^2 - \frac{1}{4}x^2} \Rightarrow A(x) = \frac{1}{2}x\sqrt{(5 - \frac{1}{2}x)^2 - \frac{1}{4}x^2} = \frac{1}{2}x\sqrt{25 - 5x}, \quad 0 \leq x \leq 5.$$

(The upper limit of 5 is clear both because a value of $x$ larger than this would put a negative number under the radical, and because the single side cannot be more than half the perimeter.) In order to maximize $A(x)$, we compute

$$A'(x) = \frac{\frac{25}{2} - \frac{15}{4}x}{\sqrt{25 - 5x}}.$$

The critical values are $x = \frac{10}{3}, 5$, where we observe that $x = 5$ is also a boundary value.

Checking $A(x)$ at the critical and boundary values, we find

$$
\begin{aligned}
A(0) &= 0 \\
A(\frac{10}{3}) &= \frac{5}{3}\sqrt{\frac{25}{3}} = \frac{25}{3\sqrt{3}} \\
A(5) &= 0.
\end{aligned}
$$

We conclude that the maximum area is $\frac{25}{3\sqrt{3}}$ and the side-lengths are $x = \frac{10}{3}$ and $y = 5 - \frac{1}{2}(\frac{10}{3}) = \frac{10}{3}$. That is, an equilateral triangle.

13. First, the fixed points are solutions of

$$x = \frac{2}{1+x} \Rightarrow x(1+x) = 2 \Rightarrow x^2 + x - 2 = 0.$$

Factoring, we find $x = -2, 1$. The graph of

$$f(x) = \frac{2}{1+x},$$

along with the cobweb is given in Figure 7. We condclude

$$\lim_{t\to\infty} x_t = 1.$$

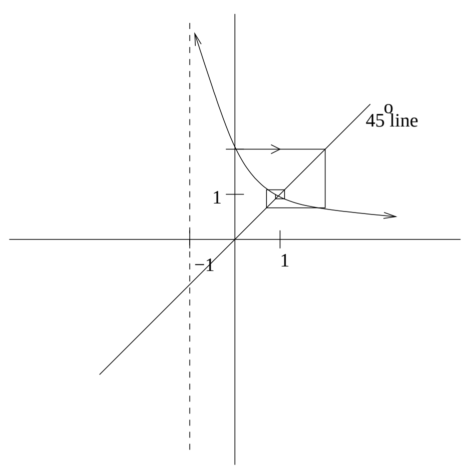


Figure 7: Figure for Problem 13 solution.

14. The fixed points are solutions of

$$x = xe^{\frac{1}{2}(1-x^3)}.$$

We see that $x = 0$ is a fixed point, and the other fixed point is a solution of

$$1 = e^{\frac{1}{2}(1-x^3)} \Rightarrow 0 = \frac{1}{2}(1-x^3) \Rightarrow x = 1.$$

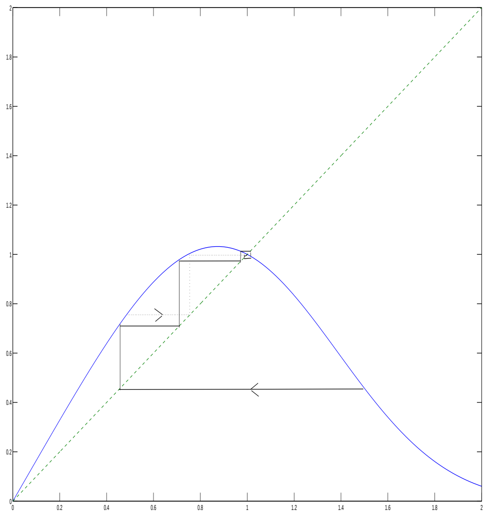

Figure 8: Figure for Problem 14 solution.

The cobweb graph is shown in Figure 8. (The cobweb starting at $x_0 = \frac{1}{2}$ has been drawn with a dashed line.) We see that in either case (i.e., $x_0 = \frac{1}{2}$ or $x_0 = \frac{3}{2}$) we have $\lim_{t\to\infty} x_t = 1$.

15. In order to find the fixed points, we solve

$$x = 1 + \frac{2}{x},$$

which becomes (upon multiplication by $x$)

$$x^2 - x - 2 = (x-2)(x+1) = 0,$$

and the fixed points are $x = -1, 2$. In order to check for stability we set $f(x) = 1 + \frac{2}{x}$, and compute

$$f'(x) = -\frac{2}{x^2}.$$

We have

$$f'(-1) = -2 \Rightarrow -1 \text{ is unstable}$$
$$f'(2) = -\frac{1}{2} \Rightarrow 2 \text{ is stable.}$$

16. The graph of the function $f(x) = |x|$ looks like a V on $[-1, 1]$, and the area under the curve consists of two triangles with equal areas. (See Figure 9.) Each triangle has baselength 1 and height 1, and so the area of each is $\frac{1}{2}$. We conclude

$$\int_{-1}^{1} |x| dx = 1.$$

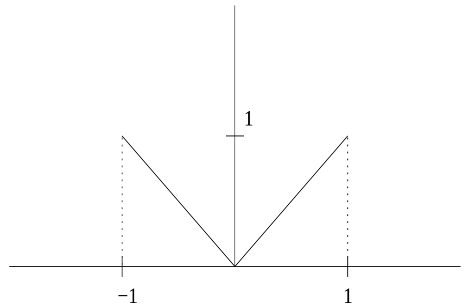


Figure 9: Figure for Problem 16 solution.

17. Since no value for $f(x)$ is given at either $x = 0$ or at $x = 1$, we cannot take a Riemann sum with left or right endpoints. We see, however, that the values of $x$ are precisely the midpoints of the subintervals in the partition $P = [0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1]$. The most reasonable Riemann sum is

$$\sum_{k=1}^{4} f(c_k)\Delta x_k,$$

where the $c_k$ are the interval midpoints. That is,

$$\sum_{k=1}^{4} f(c_k)\Delta x_k = (\frac{1}{2} + \frac{1}{3} - 1 - 2)\frac{1}{4} = -\frac{13}{24}.$$

18. In this case $\triangle x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$, and we use right endpoints $x_k = 1 + k\triangle x$. We have

$$\begin{aligned}
A_n &= \sum_{k=1}^{n}[(1 + k\Delta x) + (1 + k\Delta x)^2]\Delta x \\
&= \sum_{k=1}^{n}[(1 + \frac{k}{n}) + (1 + 2\frac{k}{n} + \frac{k^2}{n^2})]\frac{1}{n} \\
&= \Big[\frac{1}{n}\sum_{k=1}^{n} 1 + \frac{1}{n^2}\sum_{k=1}^{n} k + \frac{1}{n}\sum_{k=1}^{n} 1 + \frac{2}{n^2}\sum_{k=1}^{n} k + \frac{1}{n^3}\sum_{k=1}^{n} k^2\Big] \\
&= \Big[1 + \frac{1}{n^2}\frac{n(n+1)}{2} + 1 + \frac{2}{n^2}\frac{n(n+1)}{2} + \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\Big].
\end{aligned}$$

Finally,

$$\int_1^2 x + x^2 dx = \lim_{n\to\infty} A_n = 1 + \frac{1}{2} + 1 + 1 + \frac{1}{3} = \frac{23}{6}.$$

19. In this case $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$, and we use right endpoints $x_k = 1 + k\Delta x$. We have

$$\begin{aligned} A_n &= \sum_{k=1}^{n} [1 - (1 + \frac{3k}{n})^2] \frac{3}{n} = \sum_{k=1}^{n} [1 - (1 + 6\frac{k}{n} + 9\frac{k^2}{n^2})] \frac{3}{n} \\ &= \sum_{k=1}^{n} -\frac{18k}{n^2} - \frac{27k^2}{n^3} = -\frac{18}{n^2} \sum_{k=1}^{n} k - \frac{27}{n^3} \sum_{k=1}^{n} k^2 \\ &= -\frac{18}{n^2} \frac{n(n+1)}{2} - \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

We conclude

$$\int_1^4 1 - x^2 dx = \lim_{n\to\infty} A_n = -9 - 9 = -18.$$

20. This function is continuous on the interval $[0, 2]$ and so FTC applies. In order to compute the anti-derivative, we first observe that for $x \in [0, 1]$ we have

$$F(x) = \int_0^x y dy = \frac{x^2}{2},$$

as expected. For $x \in [1, 2]$ we must keep in mind that we have

$$F(x) = \int_0^x f(y)dy = \int_0^1 y dy + \int_1^x 2 - y dy = \frac{1}{2} + (2x - \frac{x^2}{2}) - \frac{3}{2} = (2x - \frac{x^2}{2}) - 1.$$

That is,

$$F(x) = \begin{cases} \frac{x^2}{2} & 0 \le x \le 1 \\ (2x - \frac{x^2}{2}) - 1 & 1 < x \le 2 \end{cases}.$$

Applying FTC, we conclude

$$\int_0^2 f(x)dx = F(2) - F(0) = 1.$$

(This can easily be verified by a geometric argument.)

21. According to Leibniz' rule, we have

$$\frac{d}{dx} \int_{\cos x}^{x^2+1} e^{-x^2} dx = e^{-(x^2+1)^2} 2x - e^{-\cos^2 x}(-\sin x).$$

22a. Using the substitution $u = e^x$, for which $du = e^x dx$, we find

$$\int \cos u du = \sin u + C = \sin(e^x) + C.$$

22b. We make the substitution $u = 1 + x$, with $du = dx$, and obtain

$$\begin{aligned}\int \frac{x}{\sqrt{u}} du = \int \frac{u-1}{\sqrt{u}} du = \int u^{1/2} - u^{-1/2} du \\ = \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + C = \frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C \\ = (1+x)^{1/2}(\frac{2}{3}x - \frac{4}{3}) + C.\end{aligned}$$

22c. Use the substitution $u = 2x - 1$, so that $\frac{du}{dx} = 2$. The integral becomes

$$\frac{1}{2}\int \cos u du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(2x-1) + C.$$

22d. Use the substitution $u = x^2 + 1$, so that $\frac{du}{dx} = 2x$. The integral becomes

$$\int \frac{x}{u}\frac{du}{2x} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C,$$

where since $x^2 + 1$ is always positive the absolute value can be dropped.

23a. Use the substitution $u = x^2$, so that $\frac{du}{dx} = 2x$. The integral becomes

$$\int_0^\pi x\sin(u)\frac{du}{2x} = \frac{1}{2}\int_0^\pi \sin u du = -\frac{1}{2}\cos u\Big|_0^\pi = 1.$$

23b. Use the substitution $u = \ln x$, so that $\frac{du}{dx} = \frac{1}{x}$. The integral becomes

$$\int_1^e \frac{\sqrt{u}}{x} x du = \int_0^1 u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}.$$

23c. We proceed by writing $\sin|x|$ as

$$\sin|x| = \begin{cases} -\sin x, & -\frac{\pi}{2} \le x \le 0 \\ \sin x, & 0 < x \le \pi \end{cases}.$$

In this way, we can compute

$$\int_{-\frac{\pi}{2}}^{\pi} \sin|x| dx = -\int_{-\frac{\pi}{2}}^{0} \sin x dx + \int_0^\pi \sin x dx = \cos x\Big|_{-\frac{\pi}{2}}^{0} - \cos x\Big|_0^\pi = 1 - (-1 - 1) = 3.$$

23d. We make the substitution $u = 1 + x^3$ (or alternatively use fast substitution), so that $du = 3x^2 dx$, and the integral becomes

$$\int_2^{28} \frac{x^2}{\sqrt{u}}\frac{du}{3x^2} = \frac{1}{3}\int_2^{28} u^{-1/2} du = \frac{1}{3}\frac{u^{1/2}}{1/2}\Big|_2^{28} = \frac{2}{3}[\sqrt{28} - \sqrt{2}].$$

24. We make the substitution $u = 1 + \sin^2 x$, with $du = 2\sin x \cos x dx$, and we find

$$\int \frac{\sin^3 x \cos x}{\sqrt{u}} \frac{du}{2\sin x \cos x} = \frac{1}{2}\int \frac{\sin^2 x}{\sqrt{u}} du.$$

At this point we observe that $\sin^2 x = u - 1$, so we have

$$\frac{1}{2}\int \frac{u-1}{\sqrt{u}} du = \frac{1}{2}\int u^{1/2} - u^{-1/2} du = \frac{1}{2}[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}] = \frac{1}{3}(1+\sin^2 x)^{3/2} - (1+\sin^2 x)^{1/2} + C.$$