# M151B Practice Problems for Exam 1

Calculators will not be allowed on the exam. Unjustified answers will not receive credit.

1. Compute each of the following limits:

1a.
$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}.$$

1b.
$$\lim_{x \to 3^-} \frac{x}{x^2 - 2x - 3}.$$

1c.
$$\lim_{x \to 0} \frac{\sin 7x}{x}.$$

1d.
$$\lim_{x \to 1} \frac{\sqrt{x^2 + 1} - \sqrt{x + 1}}{x - 1}.$$

1e.
$$\lim_{x \to -\infty} \frac{x^3 - x^2 + 1}{1 - x^2}.$$

1f.
$$\lim_{x \to \infty} (e^{-x} \sin x).$$

2. Find all points at which
$$\frac{\ln(1 - x)}{\ln(1 + x)}$$
is continuous.

3. Find a value for $c$ that makes the given function continuous at all points.
$$f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ x - c, & x > 1 \end{cases}.$$

4. Prove that the equation
$$e^x - 2 = \sin x$$
has at least one real-valued solution.

5. Use the bisection method to approximate a root of
$$x^4 + x^3 + x - 1 = 0$$
with a maximum error of $\frac{1}{3}$.

6. Use the definition of limits to prove the following statement:

$$\lim_{x \to 2} 7x - 1 = 13.$$

7. Use the definition of limits to prove the following statement:

$$\lim_{x \to 3} (x^2 + 1) = 10.$$

8. Use the definition of derivative to compute the derivative of the following function at $x = 0$.

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

9. Determine whether or not each of the following functions is differentiable at the point $x = 0$. In each case, explain why or why not.

9a.

$$f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ x^2 - 1, & x > 0 \end{cases}.$$

9b.

$$f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ 2x + 1, & x > 0 \end{cases}.$$

9c.

$$f(x) = x|x|.$$

10. Find an equation for the line that is tangent to the given curve at $x = 1$.

$$y = x^3 + 1.$$

Sketch a graph of the curve along with this tangent line.

11. Compute the derivative of each of the following functions:

11a.

$$f(x) = x^{\frac{2}{3}} + x^{-7}.$$

11b.

$$f(x) = x \sin x.$$

11c.

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

11d.

$$f(x) = (2x + \frac{1}{x})^2.$$

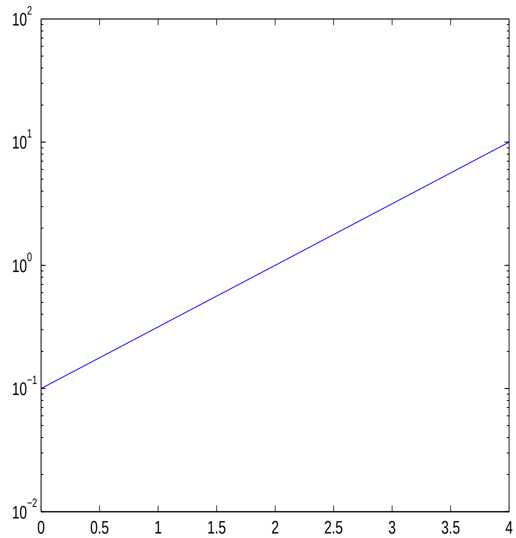


Figure 1: Figure for Problem 21.

12. Compute $f''(x)$ if

$$f(x) = \sin(\sqrt{2^x}).$$

13. Compute $\frac{dy}{dx}$ given that

$$\sin(xy) = x.$$

Find an equation for the line that is tangent to this curve at the point $(\frac{1}{\sqrt{2}}, \frac{\sqrt{2}\pi}{4})$.

14. An airplane is flying 6 miles above the ground on a flight path that will take it directly over a radar tracking station. If the distance between the plane and tracking station is decreasing at a rate of 400 miles per hour when the distance is 10 miles, what is the speed of the plane?

15. Suppose that two sides of a triangle are 4 cm and 5 cm in length and that the angle between them is increasing at a rate of .06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides is $\frac{\pi}{3}$.

16. Use a logarithmic transformation to find a linear relationship between (appropriate transformations of) $x$ and $y$ if

$$y = 2 \times 7^{4x}.$$

17. Given the double-log plot in Figure 1, find a functional relationship between $x$ and $y$.

# Solutions

1a. Compute

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

Notice in particular that we don't have to be able to evaluate the function at a point to compute its limit at that point.

1b. Compute

$$\lim_{x\to 3^-} \frac{x}{x^2-2x-3} = \lim_{x\to 3^-} \frac{x}{(x-3)(x+1)} = -\infty.$$

1c. We make the substitution $y = 7x$, and our limit becomes

$$\lim_{y\to 0} \frac{\sin y}{(y/7)} = 7 \lim_{y\to 0} \frac{\sin y}{y} = 7.$$

You won't lose points on a problem like this if you omit the explicit substitution.

1d. In this case, we rationalize the numerator,

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{x^2+1}-\sqrt{x+1}}{x-1} &= \lim_{x\to 1} \frac{\sqrt{x^2+1}-\sqrt{x+1}}{x-1} \cdot \frac{\sqrt{x^2+1}+\sqrt{x+1}}{\sqrt{x^2+1}+\sqrt{x+1}} \\
&= \lim_{x\to 1} \frac{x^2+1-(x+1)}{(x-1)(\sqrt{x^2+1}+\sqrt{x+1})} \\
&= \lim_{x\to 1} \frac{x^2-x}{(x-1)(\sqrt{x^2+1}+\sqrt{x+1})} = \lim_{x\to 1} \frac{x(x-1)}{(x-1)(\sqrt{x^2+1}+\sqrt{x+1})} \\
&= \lim_{x\to 1} \frac{x}{(\sqrt{x^2+1}+\sqrt{x+1})} = \frac{1}{2\sqrt{2}}.
\end{aligned}$$

1e. According to our rule from class, the following calculation is entirely fair:

$$\lim_{x\to -\infty} \frac{x^3-x^2+1}{1-x^2} = \lim_{x\to -\infty} \frac{x^3}{-x^2} = \lim_{x\to -\infty} (-x) = +\infty.$$

1f. Since $\sin x$ does not have a limit as $x \to \infty$ we use the Squeeze Theorem (a.k.a. the Sandwich Theorem), observing

$$-e^{-x} \leq e^{-x} \sin x \leq e^{-x}.$$

We have $\lim_{x\to\infty}(-e^{-x}) = \lim_{x\to\infty}(e^{-x}) = 0$, so by the Squeeze Theorem

$$\lim_{x\to\infty} e^{-x} \sin x = 0.$$

2. First, observe that $\ln(1-x)$ is only defined for $x < 1$ and $\ln(1+x)$ is only defined for $x > -1$, so our range is restricted to this interval. Also, we cannot divide by 0, so we must have $x \neq 0$. We conclude that the points of continuity are

$$(-1, 0) \cup (0, 1).$$

3. We observe that the only point at which $f$ may not be continuous is $x = 1$, and at this point $f(1) = 2$. In order to make the function continuous at this point, we must ensure

$$\lim_{x\to 1^+} x - c = 2,$$

and this requires $c = -1$.

4. We begin by defining the function

$$f(x) = e^x - 2 - \sin x,$$

and we note that our goal will be to show that $f(x)$ has at least one real root. First, we observe that $f(0) = -1$. Next, we observe that since $e > 2$ we know that $e^2 > 4$, so that $f(2) = e^2 - 2 - \sin 2 > 0$. We can conclude from the Intermediate Value Theorem that there is a root on the interval $(0, 2)$.

5. We begin by defining the function

$$f(x) = x^4 + x^3 + x - 1,$$

and we observe that $f(0) = -1$ and $f(1) = 2$, so that we are guaranteed a root in $(0, 1)$. We take

$$c_1 = \frac{1}{2} \pm \frac{1}{2},$$

and compute

$$f(\frac{1}{2}) = (\frac{1}{2})^4 + (\frac{1}{2})^3 + \frac{1}{2} - 1 = \frac{1}{16} + \frac{1}{8} + \frac{1}{2} - 1 = \frac{1 + 2 + 8 - 16}{16} = -\frac{5}{16} < 0.$$

We conclude that the root is on the interval $(\frac{1}{2}, 1)$, and our second approximation becomes

$$c_2 = \frac{\frac{1}{2} + 1}{2} = \frac{3}{4} \pm \frac{1}{4},$$

and $\frac{1}{4} < \frac{1}{3}$, so this is a sufficient approximation. (**Note.** This equation has a second real root between -2 and -1, so it's possible to approximate that one instead, which is fine.)

6. Our goal will be to show that

$$|(7x - 1) - 13| < \epsilon.$$

For linear functions we choose $\delta = \frac{\epsilon}{7}$. We now verify that this works by computing

$$|(7x - 1) - 13| = |7x - 14| = 7|x - 2| < 7\delta = 7\frac{\epsilon}{7} = \epsilon.$$

This completes the proof. □

7. In this case we want to show

$$0 < |x - 3| < \delta \implies |(x^2 + 1) - 10| < \epsilon.$$

As usual, we begin with $|(x^2 + 1) - 10|$, and in this case we compute

$$|(x^2 + 1) - 10| = |x^2 - 9| = |(x + 3)(x - 3)| = |x + 3||x - 3|.$$

As our first choice of $\delta$, we take $\delta = 1$, so that

$$2 < x < 4,$$

and we obtain the inequality $|x+3| < 7$. (Note that in this case any positive value for $\delta$ works fine.) We now have

$$|x+3||x-3| < 7|x-3| < 7\delta,$$

and we take as our second choice of $\delta$ $\delta = \frac{\epsilon}{7}$. Thus $\delta = \min\{1, \frac{\epsilon}{7}\}$. For Step 2, we compute

$$|(x^2+1) - 10| = |x+3||x-3| \overset{\text{first choice}}{<} 7|x-3| < 7\delta \overset{\text{second choice}}{\leq} \epsilon,$$

where as mentioned in class it's fine with me if the final inequality is written as equality since $\frac{\epsilon}{7}$ is generally smaller than 1.

8. According to the definition,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \cos \frac{1}{h}}{h} = \lim_{h\to 0} h \cos \frac{1}{h}.$$

Observing now that $|\cos \frac{1}{h}| \leq 1$, we have

$$-|h| \leq h \cos \frac{1}{h} \leq |h|,$$

and so by the squeeze theorem $f'(0) = 0$. Notice that with this definition of $f(x)$, we cannot compute the derivative at $x = 0$ from our formulas. That is,

$$f'(x) = 2x \cos \frac{1}{x} - \sin \frac{1}{x},$$

which is not defined at $x = 0$ and has no limit as $x \to 0$. (This is a case in which $f(x)$ is differentiable at a point, but $f'(x)$ is not continuous at that point.)

9a. We can see that $f(x)$ is not continuous at $x = 0$ by computing

$$\lim_{x\to 0^-} f(x) = 1,$$

and

$$\lim_{x\to 0^+} f(x) = -1,$$

from which we conclude that the limit of $f(x)$ as $x \to 0$ does not exist. We know that if a function is not continuous at a point then it cannot be differentiable at that point, so $f(x)$ is not differentiable at $x = 0$.

9b. **Method 1.** According to the definition of derivative,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

Since $f$ is defined by different functions for $h < 0$ and for $h > 0$, we must compute and compare right and left limits:

$$\lim_{h\to 0^-} \frac{(h^2+1) - 1}{h} = \lim_{h\to 0^-} h = 0,$$

while

$$\lim_{h\to 0^+} \frac{(2h+1)-1}{h} = \lim_{h\to 0^+} 2 = 2.$$

Since these limits do not agree, we can conclude that $f(x)$ is not differentiable at $x = 0$.

**Method 2.** In cases for which

$$f(x) = \begin{cases} f_1(x) & x \le a \\ f_2(x) & x > a \end{cases},$$

where $f_1(a) = f_2(a)$, $f_1'(a) = c_1$, and $f_2'(a) = c_2$ we can proceed as follows: if $c_1 \neq c_2$ then $f$ is not differentiable at $x = a$, while if $c_1 = c_2$ then $f$ is differentiable at $x = a$ and $f'(a) = c_1$. Here, $f_1(x) = x^2 + 1$ and $f_2(x) = 2x + 1$, so $f'(0) = 0$ while $f_2'(0) = 2$. We can draw the same conclusion as we did with Method 1. (Be sure to check all assumptions when using this method; try it, for example, on (2a).)

9c. In this case,

$$f'(0) = \lim_{h\to 0} \frac{h|h|}{h} = \lim_{h\to 0} |h| = 0,$$

and so $f$ is differentiable at $x = 0$ with $f'(0) = 0$.

10. The slope of the tangent line is given by the derivative $y'(1) = 3(1)^2 = 3$. We have, then $y - 2 = 3(x - 1)$.

11.

11a. Applying the power rule to each summand, we find

$$\frac{d}{dx}(x^{\frac{2}{3}} + x^{-7}) = \frac{2}{3}x^{-\frac{1}{3}} - 7x^{-8}.$$

11b. Applying the product rule, we find

$$\frac{d}{dx} x \sin x = \sin x + x \cos x.$$

11c. Applying the quotient rule, we find

$$\frac{d}{dx} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2},$$

and though considerable simplification is possible, this form is sufficient for the exam.

11d. Proceeding with the chain rule, we set $u = 2x + \frac{1}{x}$ and compute

$$\frac{d}{dx}u^2 = \frac{d}{du}u^2\frac{du}{dx} = 2u(2 - \frac{1}{x^2}) = 2(2x + \frac{1}{x})(2 - \frac{1}{x^2}).$$

(This substitution does not need to be made explicitly.)

12. **Method 1.** Compute directly

$$f'(x) = \cos(\sqrt{2^x})\frac{1}{2\sqrt{2^x}}2^x \ln 2 = \frac{\ln 2}{2}\cos(\sqrt{2^x})\sqrt{2^x},$$

and

$$\begin{aligned} f''(x) &= \frac{\ln 2}{2}\Big( - \sin(\sqrt{2^x})\frac{\ln 2}{2}2^x + \cos(\sqrt{2^x})\frac{\ln 2}{2}\sqrt{2^x}\Big) \\ &= (\frac{\ln 2}{2})^2\Big(\sqrt{2^x}\cos(\sqrt{2^x}) - 2^x\sin(\sqrt{2^x}\Big). \end{aligned}$$

**Method 2.** First, observe that $\sqrt{2^x} = (\sqrt{2})^x$, which eliminates the need for a nested chain rule. Now,

$$\frac{d}{dx}\sin((\sqrt{2})^x) = \cos((\sqrt{2})^x)(\sqrt{2})^x \ln\sqrt{2},$$

and

$$\begin{aligned} \frac{d^2}{dx^2}\sin((\sqrt{2})^x) &= -\sin((\sqrt{2})^x)((\sqrt{2})^x\ln\sqrt{2})^2 + \cos((\sqrt{2})^x)((\sqrt{2})^x\ln\sqrt{2})\ln\sqrt{2} \\ &= (\ln\sqrt{2})^2\Big(\cos((\sqrt{2})^x)(\sqrt{2})^x - \sin((\sqrt{2})^x)2^x\Big), \end{aligned}$$

which is equivalent to the expression from Method 1.

13. We compute implicitly

$$\frac{d}{dx}\sin(xy) = \frac{d}{dx}x \Rightarrow \cos(xy)\frac{d}{dx}(xy) = 1 \Rightarrow \cos(xy)(y + x\frac{dy}{dx}) = 1.$$

Solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{\frac{1}{\cos(xy)} - y}{x}.$$

At the point $(\frac{1}{\sqrt{2}}, \frac{\sqrt{2}\pi}{4})$, we have

$$\frac{dy}{dx} = \frac{\frac{1}{\cos(\frac{\pi}{4})} - \frac{\sqrt{2}\pi}{4}}{\frac{1}{\sqrt{2}}} = 2 - \frac{\pi}{2}.$$

The equation for the tangent line is

$$(y - \frac{\sqrt{2}\pi}{4}) = (2 - \frac{\pi}{2})(x - \frac{1}{\sqrt{2}}).$$

14. In this case, we are given that $\frac{dr}{dt} = -400$, where $r$ denotes the distance between the plane and the tracking station. If we let $x$ denote the horizontal distance between the plane and the tracking station, then what we are looking for is $|\frac{dx}{dt}|$, the plane's speed. In order to find a relation between $\frac{dx}{dt}$ and $\frac{dr}{dt}$, we begin by relating $x$ and $r$. We have

$$x^2 + 36 = r^2.$$

Upon differentiation of this equation with respect to $t$, we find

$$2x\frac{dx}{dt} = 2r\frac{dr}{dt}.$$

When $r = 10$, we have $x = \sqrt{100 - 36} = 8$, and therefore

$$2(8)\frac{dx}{dt} = 2(10)(-400) \Rightarrow \frac{dx}{dt} = -\frac{8000}{16} = -500.$$

The negative sign indicates that the plane is moving toward the tracking station, but since the problem asks for *speed*, the correct answer is $+500$.

15. First, observe that what we know is $\frac{d\theta}{dt}$ and what we would like to know is $\frac{dA}{dt}$, where $A$ is the area of the triangle and $\theta$ is the angle between the sides of lengths 4 and 5. In order to get a relationship between $A$ and $\theta$, we recall that the area of a triangle is

$$A = \frac{1}{2}bh,$$

where $b$ is the length of the base of the triangle and $h$ is the height of the triangle. If we draw the triangle with the 5 cm side as the base (i.e., $b = 5$) and let $h$ denote the triangle's height, then we immediately find the relation

$$\sin\theta = \frac{h}{4} \Rightarrow h = 4\sin\theta.$$

We can now write the area as

$$A = \frac{1}{2}(5)4\sin\theta = 10\sin\theta.$$

In order to get a relationship between $\frac{dA}{dt}$ and $\frac{d\theta}{dt}$, we take the derivative of each side of this last expression with respect to $t$. We find

$$\frac{dA}{dt} = 10\cos\theta\frac{d\theta}{dt}.$$

At $\theta = \frac{\pi}{3}$ $(60^o)$, we have $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and consequently

$$\frac{dA}{dt} = 10(\frac{1}{2})(.06) = .3 \text{ cm}^2/s.$$

16. You can proceed by taking a logarithm of this equation to any base. Though 7 would be a reasonable base here, it is sufficiently uncommon that I'll use base 10. That is,

$$\log_{10} y = \log_{10} 2 \times 7^{4x} \Rightarrow \log_{10} y = \log_{10} 2 + 4x\log_{10} 7.$$

The linear relationship is

$$\log_{10} y = (4 \log_{10} 7)x + \log_{10} 2.$$

17. This is a semilog plot with $\log_{10} y$ on the vertical axis and $x$ on the horizontal. That is, the line has an equation of the form

$$\log_{10} y = mx + b.$$

We can read directly from the plot that $b = -1$. Likewise, the slope is

$$m = \frac{1 - (-1)}{4} = \frac{1}{2}.$$

We have, then,

$$\log_{10} y = \frac{1}{2}x - 1.$$

In order to get a functional relationship, exponentiate each side with base 10,

$$10^{\log_{10} y} = 10^{\frac{1}{2}x - 1} \Rightarrow y = 10^{-1}(10^{\frac{1}{2}})^x.$$