M151B Practice Problems for Exam 3

Calculators will not be allowed on the exam. On the exam you will be given the following identities:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \qquad \sum_{k=1}^{n} k^3 = \Big(\frac{n(n+1)}{2}\Big)^2.$$

1. Write down a general expression for the sequence with terms

$$\frac{3}{2}, -\frac{5}{8}, \frac{7}{18}, -\frac{9}{32}, \frac{11}{50}, \dots$$

2. Find all fixed points for the recursion

$$a_{n+1} = \frac{1}{2}a_n\left(\frac{1}{2} - a_n\right),$$

and use the method of cobwebbing to determine which limit will be achieved from the starting value $a_0 = -1$.

3. Find all fixed points for the recursion

$$x_{t+1} = x_t e^{1-x_t},$$

and determine whether each is asymptotically stable or unstable.

4. The discrete logistic population model is

$$N_{t+1} = N_t + RN_t\left(1 - \frac{N_t}{K}\right).$$

Take $R = 1$ and $K = 10$ and show that one drawback of this model is that it can start with a positive population $N_0 > 0$ and return a negative population N_1.

5. Use a geometric argument to evaluate the integral

$$\int_{-1}^{1} |x| dx.$$

6. Use the method of Riemann sums to evaluate

$$\int_{1}^{4} 1 - x^2 dx.$$

7. Express the integral

$$\int_{0}^{2} \sqrt{1 + \sin x}\, dx$$

as the limit of a Riemann sum. Be sure to define all quantities that appear in your expression.

8. Determine whether or not the Fundamental Theorem of Calculus can be applied to the function

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 < x \leq 2 \end{cases},$$

on the interval $[0, 2]$. If so, find the anti-derivative $F(x)$ and use it to compute

$$\int_0^2 f(x)dx.$$

9. Compute

$$\frac{d}{dx} \int_{\cos x}^{x^2+1} e^{-x^2} dx.$$

10. Evaluate the following indefinite integrals.

10a.

$$\int \cos(2x - 1)dx.$$

10b.

$$\int \frac{x}{x^2 + 1} dx.$$

11. Evaluate the following definite integrals.

11a.

$$\int_0^{\sqrt{\pi}} x \sin(x^2)dx.$$

11b.

$$\int_1^e \frac{\sqrt{\ln x}}{x} dx.$$

12. Evaluate the following indefinite integral

$$\int \frac{\sin^3 x \cos x}{\sqrt{1 + \sin^2 x}} dx.$$

13. Find the total area between the curves $y = x^2$ and $y = 2 - x$ for $x \in [0, 2]$.

14. Suppose that for each value x a certain solid has cross-sectional area $A(x)$ and density $\rho(x)$. (The density of an object is its mass per unit volume, $\rho = \frac{m}{V}$.) Find a formula for the total mass of such an object on the interval $[a, b]$.

15. Find the volume of the solid obtained by rotating about the x-axis the area between the graph of $f(x) = \sqrt{x}$ and the x-axis for $x \in [0, 1]$.

16. Find the volume obtained by rotating the region between $y = 2$ and $y = \sqrt{x}$ for $x \in [0, 4]$ about the x-axis.

17. Find the volume obtained by rotating the region bounded by the curves $y^2 = x$ and $y = \frac{x}{2}$ about the y-axis.

18. Suppose the base of a certain solid is the region in the xy-plane bounded by $y = 4$ and $y = x^2$. Find the volume of the solid created if every cross section is a square perpendicular to the x-axis.

19. Compute the average value of the function

$$f(x) = x + \frac{1}{x}$$

for $x \in [1, 3]$.

20. Determine the length of the graph of $f(x) = x^{\frac{3}{2}} + 1$ for $x \in [0, 4]$.

Solutions.

1. First, we get the sign right with $(-1)^{n+1}$, $n = 1, 2, \ldots$, and we observe that the numerator is $2n + 1$, for $n = 1, 2, \ldots$. The easiest way to understand the denominator is to factor out the common factor 2 (a useful trick in general). We find

$$a_n = (-1)^{n+1} \frac{2n+1}{2n^2}, \quad n = 1, 2, \ldots$$

2. First, the fixed point equation is

$$a = \frac{1}{2}a(\frac{1}{2} - a) = \frac{1}{4}a - \frac{1}{2}a^2,$$

so that the fixed points are

$$\frac{3}{4}a + \frac{1}{2}a^2 = a(\frac{3}{4} + \frac{1}{2}a) = 0 \Rightarrow a = 0, -\frac{3}{2}.$$

For the cobwebbing, we can plot $f(a) = \frac{1}{4}a - \frac{1}{2}a^2 = a(\frac{1}{4} - \frac{1}{2}a)$ by noticing that it's a parabola opening downward with x-intercepts at $a = 0$ and $a = \frac{1}{2}$, and therefore has a maximum value at (the midpoint) $\frac{1}{4}$ of $\frac{1}{4}(\frac{1}{4} - \frac{1}{2}\frac{1}{4}) = \frac{1}{4}\frac{1}{8} = \frac{1}{32}$. We find that for $a_0 = -1$ (see the figure)

$$\lim_{n \to \infty} a_n = 0.$$

3. First, in order to find the fixed points we solve

$$x = xe^{1-x} \Rightarrow x(1 - e^{1-x}) = 0,$$

from which we have the fixed points

$$x = 0, 1.$$

In order to evaluate the stability of these points, we set

$$f(x) = xe^{1-x},$$

and compute

$$f'(x) = e^{1-x} + xe^{1-x}(-1) = e^{1-x}(1-x).$$

We have:

$$\begin{aligned}
f'(0) &= e \Rightarrow |f'(0)| > 1 \Rightarrow x = 0 \text{ is unstable}\\
f'(1) &= 0 \Rightarrow |f'(0)| < 1 \Rightarrow x = 1 \text{ is asymptotically stable}
\end{aligned}$$

4. First, for $R = 1$ and $K = 10$ the model becomes

$$N_{t+1} = N_t + N_t(1 - \frac{N_t}{10}).$$

We see that if N_t is large the second term will be negative, and as a convenient value we can take $N_0 = 50$. We find

$$N_1 = 50 + 50(1 - 5) = 50 - 200 = -150.$$

5. The graph of the function $f(x) = |x|$ looks like a V on $[-1, 1]$, and the area under the curve consists of two triangles with equal areas. Each triangle has baselength 1 and height 1, and so the area of each is $\frac{1}{2}$. We conclude

$$\int_{-1}^{1} |x| dx = 1.$$

6. In this case $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$, and we use right endpoints $x_k = 1 + k\Delta x$. We have

$$\begin{aligned}
A_n &= \sum_{k=1}^{n} [1 - (1 + \frac{3k}{n})^2] \frac{3}{n} = \sum_{k=1}^{n} [1 - (1 + 6\frac{k}{n} + 9\frac{k^2}{n^2})] \frac{3}{n}\\
&= \sum_{k=1}^{n} -\frac{18k}{n^2} - \frac{27k^2}{n^3} = -\frac{18}{n^2} \sum_{k=1}^{n} k - \frac{27}{n^3} \sum_{k=1}^{n} k^2\\
&= -\frac{18}{n^2} \frac{n(n+1)}{2} - \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

We conclude

$$\lim_{n \to \infty} A_n = -9 - 9 = -18.$$

7. The Riemann sum is

$$\int_0^2 \sqrt{1+\sin x}dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{1+\sin c_k}\triangle x_k,$$

where P is a partition of the interval $[0,2]$ with points $P=[x_0, x_1, ..., x_n]$, $\|P\|$ is the norm of P ($\|P\| = \max_k \Delta x_k$), $\Delta x_k = x_k - x_{k-1}$, and $c_k \in [x_{k-1}, x_k]$ for each $k=1,...,n$.

8. This function is continuous on the interval $[0,2]$ and so FTC applies. In order to compute the anti-derivative, we first observe that for $x \in [0,1]$ we have

$$F(x) = \int_0^x ydy = \frac{x^2}{2},$$

as expected. For $x \in [1,2]$ we must keep in mind that we have

$$F(x) = \int_0^x f(y)dy = \int_0^1 ydy + \int_1^x 2-ydy = \frac{1}{2} + (2x - \frac{x^2}{2}) - \frac{3}{2} = (2x - \frac{x^2}{2}) - 1.$$

That is,

$$F(x) = \begin{cases} \frac{x^2}{2} & 0 \le x \le 1 \\ (2x - \frac{x^2}{2}) - 1 & 1 < x \le 2 \end{cases}.$$

Applying FTC, we conclude

$$\int_0^2 f(x)dx = F(2) - F(0) = 1.$$

(This can easily be verified by a geometric argument.)

9. According to Leibniz' rule, we have

$$\frac{d}{dx}\int_{\cos x}^{x^2+1} e^{-x^2}dx = e^{-(x^2+1)^2}2x - e^{-\cos^2 x}(-\sin x).$$

10a. Use the substitution $u = 2x-1$, so that $\frac{du}{dx} = 2$. The integral becomes

$$\frac{1}{2}\int \cos udu = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(2x-1) + C.$$

10b. Use the substitution $u = x^2+1$, so that $\frac{du}{dx} = 2x$. The integral becomes

$$\int \frac{x}{u}\frac{du}{2x} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C,$$

where since x^2+1 is always positive the absolute value can be dropped.

11a. Use the substitution $u = x^2$, so that $\frac{du}{dx} = 2x$. The integral becomes

$$\int_0^\pi x\sin(u)\frac{du}{2x} = \frac{1}{2}\int_0^\pi \sin udu = -\frac{1}{2}\cos u\Big|_0^\pi = 1.$$

11b. Use the substitution $u = \ln x$, so that $\frac{du}{dx} = \frac{1}{x}$. The integral becomes

$$\int_1^e \frac{\sqrt{u}}{x} x du = \int_0^1 u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}.$$

(Alternatively, Problems 10 and 11 can be solved with fast substitution.)

12. We make the substitution $u = 1 + \sin^2 x$, with $du = 2 \sin x \cos x dx$, and we find

$$\int \frac{\sin^3 x \cos x}{\sqrt{u}} \frac{du}{2 \sin x \cos x} = \frac{1}{2} \int \frac{\sin^2 x}{\sqrt{u}} du.$$

At this point we observe that $\sin^2 x = u - 1$, so we have

$$\frac{1}{2} \int \frac{u-1}{\sqrt{u}} du = \frac{1}{2} \int u^{1/2} - u^{-1/2} du = \frac{1}{2}[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}] = \frac{1}{3}(1 + \sin^2 x)^{3/2} - (1 + \sin^2 x)^{1/2} + C.$$

13. Plotting these two curves together, we can see that they intersect at $x = 1$, and that for $x < 1$, $y = 2 - x$ is larger, while for $x > 1$, $y = x^2$ is larger. The area between the curves is

$$\begin{aligned} A &= \int_0^1 (2 - x) - x^2 dx + \int_1^2 x^2 - (2 - x) dx \\ &= (2x - \frac{x^2}{2} - \frac{x^3}{3})\Big|_0^1 + (\frac{x^3}{3} - 2x + \frac{x^2}{2})\Big|_1^2 \\ &= 3. \end{aligned}$$

14. As when developing our formula for the volume of such an object, we consider a partition of the interval $[a, b]$, $P = [x_0, x_1, \ldots, x_n]$. On the general subinterval $[x_{k-1}, x_k]$ we approximately have a cylinder with base area $A(c_k)$ and constant density $\rho(c_k)$, where c_k is any value $c_k \in [x_{k-1}, x_k]$. The mass of such a cylinder is $\rho(c_k)V = \rho(c_k)A(c_k)\Delta x_k$, and so if we sum up these masses we have

$$M_n = \sum_{k=1}^{n} \rho(c_k)A(c_k)\Delta x_k.$$

Taking now a limit as the partition size goes to 0, we find

$$M = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \rho(c_k)A(c_k)\Delta x_k = \int_a^b \rho(x)A(x)dx.$$

15. Since the object is being created by rotation, the cross section at each point x is a circle with radius $f(x)$. The area of the cross section at point x is $A(x) = \pi f(x)^2 = \pi x$. Recalling that our volume formula is

$$V = \int_a^b A(x)dx,$$

we compute

$$V = \int_0^1 \pi x dx = \pi \frac{1}{2} x^2 \Big|_0^1 = \frac{\pi}{2}.$$

16. In this case, we use the method of washers, for which we have

$$\begin{aligned} V &= \pi \int_0^4 f(x)^2 - g(x)^2 dx \\ &= \pi \int_0^4 2^2 - (\sqrt{x})^2 dx = \pi \Big[4x - \frac{x^2}{2}\Big]_0^4 \\ &= 8\pi. \end{aligned}$$

17. First, we find the points at which these curves intersect by solving

$$(\frac{x}{2})^2 = x \Rightarrow \frac{x^2}{4} - x = 0 \Rightarrow x(\frac{x}{4} - 1) = 0 \Rightarrow x = 0, 4.$$

The points of intersection are $(0, 0)$ and $(4, 2)$. If we rotate the region between these curves about the y-axis the line $x = 2y$ describes the outer radius while the parabola $x = y^2$ describes the inner radius. The volume is

$$V = \pi \int_0^2 (2y)^2 - (y^2)^2 dy = \pi \int_0^2 4y^2 - y^4 dy = \pi[\frac{4}{3}y^3 - \frac{1}{5}y^5]_0^2 = \pi[\frac{32}{3} - \frac{32}{5}] = \frac{64\pi}{15}.$$

18. One side of each square runs from the curve $y = x^2$ to the line $y = 4$, and so the sidelength is $4 - x^2$. The cross-sectional area is consequently

$$A(x) = (4 - x^2)^2,$$

and the volume is (by symmetry)

$$V = 2\int_0^2 (4 - x^2)^2 dx = 2\int_0^2 16 - 8x^2 + x^4 dx = 2(16x - \frac{8}{3}x^3 + \frac{x^5}{5})\Big|_0^2 = \frac{512}{15}.$$

19. We compute

$$f_{\text{avg}} = \frac{1}{2}\int_1^3 x + x^{-1} dx = \frac{1}{2}[\frac{1}{2}x^2 + \ln|x|]\Big|_1^3 = \frac{1}{2}[\frac{9}{2} + \ln 3] - \frac{1}{2}[\frac{1}{2}] = 2 + \ln\sqrt{3}.$$

20. First, observe that

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}}.$$

The formula for arclength is

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx,$$

so we have

$$L = \int_0^4 \sqrt{1 + \frac{9}{4}x} dx.$$

We carry out this integral with substitution, setting $u = 1 + \frac{9}{4}x$ so that $\frac{du}{dx} = \frac{9}{4}$. The integral becomes

$$L = \int_1^{10} u^{\frac{1}{2}} \frac{4}{9} du = \frac{4}{9} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \Big|_1^{10} = \frac{8}{27}(10^{\frac{3}{2}} - 1).$$