# M151B Practice Problems for Final Exam

Calculators will not be allowed on the exam. Unjustified answers will not receive credit. On the exam you will be given the following identities:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

1. Compute each of the following limits:

1a.

$$\lim_{x\to 2^-} \frac{x}{x^2+3x-10}.$$

1b.

$$\lim_{x\to 0} x e^{\sin(\frac{1}{x})}.$$

1c.

$$\lim_{x\to 0} \frac{x\sin x}{(1-e^x)^2}.$$

1d.

$$\lim_{x\to\infty} [(x+1)^{1/3} - x^{1/3}].$$

1e. The *geometric mean* of two positive real numbers $a$ and $b$ is defined as $\sqrt{ab}$. Show that

$$\sqrt{ab} = \lim_{x\to\infty} (\frac{a^{1/x}+b^{1/x}}{2})^x.$$

2a. Find a value for $c$ that makes the given function continuous at all points.

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}.$$

2b. Determine whether or not your function from (2a) is differentiable at $x = 0$. If it is differentiable at this point, compute its derivative there.

3. Find an equation for the line that is tangent to the graph of

$$f(x) = \frac{xe^x}{1+x^2}$$

at the point $x = 0$.

4. Suppose the angle of elevation of the Sun is decreasing at a rate of .25 rad/hr. How fast is the shadow cast by a 400 ft tall building increasing when the angle of elevation of the Sun is $\frac{\pi}{6}$?

5. Suppose $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval $(a, b)$. Show that if $f'(x) = x$ for all $x \in (a, b)$, then there exists some value $c \in (0, 1)$ so that

$$f(b) - f(a) = c(b - a).$$

6. Let

$$f(x) = x^{1/3}(x + 3)^{2/3}, \quad -\infty < x < \infty.$$

6a. Locate the critical points of $f$ and determine the intervals on which $f$ is increasing and the intervals on which $f$ is decreasing.

6b. Locate the possible inflection points for $f$ and determine the intervals on which $f$ is concave up and the intervals on which it is concave down.

6c. Evaluate $f$ at the critical points and at the possible inflection points, and determine the boundary behavior of $f$ by computing limits as $x \to \pm\infty$.

6d. Use your information from Parts a-c to sketch a graph of this function.

7. Find the side-lengths that maximize the area of an isosceles triangle with given perimeter $P = 10$. (An isosceles triangle is a triangle with two sidelengths equal.)

8. Find all fixed points for the recursion equation

$$a_{n+1} = \frac{3}{4}a_n + \frac{1}{a_n}.$$

Sketch a graph of the function $f(a) = \frac{3}{4}a + \frac{1}{a}$, and use the method of cobwebbing to determine whether or not one of these fixed points will be achieved from the starting value $a_0 = \frac{1}{2}$.

9. Find all fixed points for the recursion equation

$$x_{t+1} = 1 + \frac{2}{x_t}$$

and determine whether or not each is asymptotically stable or unstable.

10. Suppose a function $f(x)$ is continuous on the interval $[0, 1]$ and that you are given the following table of values:

| $x$ | $f(x)$ |
|---|---|
| 1/8 | 1/2 |
| 3/8 | 1/3 |
| 5/8 | −1 |
| 7/8 | −2 |

Table 1: Values of $f(x)$ for Problem 1.

Use an appropriate Riemann sum to approximate $\int_0^1 f(x)dx$.

11. Use the method of Riemann sums to evaluate

$$\int_1^2 x + x^2 dx.$$

12. Evaluate the following indefinite integrals.

12a.

$$\int e^x \cos(e^x) dx.$$

12b.

$$\int \frac{x}{\sqrt{1+x}} dx$$

12b.

$$\int \cos^{-1} x dx.$$

13. Evaluate the following definite integrals.

13a.

$$\int_1^3 \frac{x^2}{\sqrt{1+x^3}} dx.$$

13b.

$$\int_0^{\frac{\pi}{4}} x \sec^2 x dx.$$

14. Find the area of the region bounded by the graphs of $y = x^4$ and $y = 20 - x^2$.

15. Find the volume obtained when the region between the graphs of $y = e^x$ and $y = e^{-x}$, $x \in [0, 2]$, is rotated about the $x$-axis.

16. Suppose the base of a certain solid is the region in the $xy$-plane between the line $y = x$ and the parabola $y = x^2$. Find the volume of the solid created if every cross section is a right isosceles triangle with hypotenuse in the $xy$-plane perpendicular to the $x$-axis.

# Solutions

1a. We have

$$\lim_{x \to 2^-} \frac{x}{x^2 + 3x - 10} = \lim_{x \to 2^-} \frac{x}{(x-2)(x+5)} = -\infty,$$

where we have observed that $x - 2$ is negative for $x$ to the left of 2.

1b. We apply the Squeeze Theorem in this case, using the inequality

$$-|x|e \leq x e^{\sin(\frac{1}{x})} \leq |x|e.$$

We have

$$\lim_{x \to 0} -|x|e = \lim_{x \to 0} |x|e = 0,$$

and so according to the Squeeze Theorem

$$\lim_{x \to 0} x e^{\sin(\frac{1}{x})} = 0.$$

1c. We apply L'Hospital's Rule twice,

$$\lim_{x\to 0}\frac{x\sin x}{(1-e^x)^2}=\lim_{x\to 0}\frac{\sin x+x\cos x}{2(1-e^x)(-e^x)}=\lim_{x\to 0}\frac{\sin x+x\cos x}{2e^{2x}-2e^x}$$
$$=\lim_{x\to 0}\frac{2\cos x-x\sin x}{4e^{2x}-2e^x}=1.$$

1d. This limit has the indeterminate form $\infty-\infty$, so the first thing we do is rearrange it into an expression with the form $\frac{0}{0}$. We have

$$\lim_{x\to\infty}(x+1)^{1/3}-x^{1/3}=\lim_{x\to\infty}x^{1/3}[(1+\frac{1}{x})^{1/3}-1]$$
$$=\lim_{x\to\infty}\frac{(1+\frac{1}{x})^{1/3}-1}{x^{-1/3}}=\lim_{x\to\infty}\frac{\frac{1}{3}(1+\frac{1}{x})^{-2/3}(-\frac{1}{x^2})}{-\frac{1}{3}x^{-4/3}}$$
$$=\lim_{x\to\infty}(1+\frac{1}{x})^{-2/3}\frac{1}{x^{2/3}}=0.$$

1e. We observe that this limit has the general form $1^\infty$, and so we can apply L'Hospital's rule. We have

$$\lim_{x\to\infty}(\frac{a^{1/x}+b^{1/x}}{2})^x=\lim_{x\to\infty}e^{\ln(\frac{a^{1/x}+b^{1/x}}{2})^x}=\lim_{x\to\infty}e^{x\ln(\frac{a^{1/x}+b^{1/x}}{2})}$$
$$=e^{\lim_{x\to\infty}x\ln(\frac{a^{1/x}+b^{1/x}}{2})}.$$

In order to compute this limit, we write

$$\lim_{x\to\infty}x\ln(\frac{a^{1/x}+b^{1/x}}{2})=\lim_{x\to\infty}\frac{\ln(\frac{a^{1/x}+b^{1/x}}{2})}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{2}{a^{1/x}+b^{1/x}}(\frac{1}{2}a^{1/x}(\ln a)(-\frac{1}{x^2})+\frac{1}{2}b^{1/x}(\ln b)(-\frac{1}{x^2}))}{-\frac{1}{x^2}}$$
$$=\lim_{x\to\infty}\frac{1}{a^{1/x}+b^{1/x}}(a^{1/x}\ln a+b^{1/x}\ln b)=\frac{1}{2}(\ln a+\ln b),$$

where in this last step we have used that $\frac{1}{x}\to 0$ as $x\to\infty$. The limit is

$$e^{\frac{1}{2}(\ln a+\ln b)}=e^{\frac{1}{2}\ln(ab)}=e^{\ln(ab)^{1/2}}=\sqrt{ab}.$$

2a. Since

$$\lim_{x\to 0}\frac{\sin x}{x}=1,$$

we can make this function continuous at all points by choosing $c=1$.

2b. Since the function is separately defined at $x=0$, we must proceed from the definition of differentiation. We compute

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\frac{\sin h}{h}-1}{h}$$
$$=\lim_{h\to 0}\frac{\sin h-h}{h^2}=\lim_{h\to 0}\frac{\cos h-1}{2h}=\lim_{h\to 0}\frac{-\sin h}{2}=0,$$

where the last two steps both used L'Hospital's rule. We conclude that this function is differentiable at $x = 0$, and that $f'(0) = 0$.

3. First,

$$f'(x) = \frac{(1+x^2)(e^x + xe^x) - xe^x(2x)}{(1+x^2)^2} \Rightarrow f'(0) = 1,$$

which is the slope of the tangent line. Using $f(0) = 0$ and the general point-slope form $y - f(a) = f'(a)(x - a)$, we conclude

$$y = x.$$

4. First, observe that what we know is $\frac{d\theta}{dt} = -.25$ rad/hr and what we want to know is $\frac{dx}{dt}$, where $x$ is the length of the shadow (see the diagram).

We see that the relation between $\theta$ and $x$ is

$$\tan\theta = \frac{400}{x}.$$

Upon taking a derivative of this equation with respect to $t$, we obtain

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{400}{x^2}\frac{dx}{dt},$$

where we can now fix $\theta = \frac{\pi}{6}$, so that $\sec^2\theta = \frac{1}{\cos^2\frac{\pi}{6}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$, while $x = \frac{400}{\tan\frac{\pi}{6}} = 400\sqrt{3}$. Combining these observations, we have

$$\frac{dx}{dt} = -\frac{x^2}{400}\frac{d\theta}{dt}\sec^2\frac{\pi}{6} = -3(400)(-.25)\frac{4}{3} = +400 \text{ ft/hr}.$$

5. According to the Mean Value Theorem there exists some value $c \in (a, b)$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

In this case $f'(c) = c$, and so we conclude

$$\frac{f(b) - f(a)}{b - a} = c \Rightarrow f(b) - f(a) = c(b - a).$$

6a. The derivative of $f(x)$ is

$$f'(x) = \frac{x+1}{x^{2/3}(x+3)^{1/3}},$$

from which we find the critical points $x = -3, -1, 0$. We see that $f$ is increasing on $(-\infty, -3] \cup [-1, \infty)$ and decreasing on $[-3, -1]$.

6b. The second derivative of $f(x)$ is

$$f''(x) = -\frac{2}{x^{5/3}(x+3)^{4/3}},$$

from which we find that the possible inflection points are $x = 0, -3$. We see that $f$ is concave up on $(-\infty, -3) \cup (-3, 0)$ and concave down on $(0, \infty)$.

6c. Evaluating $f$ at the critical points, possible inflection points, and at the endpoints, we have:

$$\begin{aligned} f(-3) &= 0 \\ f(-1) &= -2^{2/3} \\ f(0) &= 0 \\ \lim_{x\to-\infty} x^{1/3}(x+3)^{2/3} &= -\infty \\ \lim_{x\to\infty} x^{1/3}(x+3)^{2/3} &= +\infty. \end{aligned}$$

6d. Your plot should look something like this:

7. Let $y$ denote the length of the sides of equal length, and let $x$ denote the length of the side between them. Then the perimeter is

$$10 = 2y + x \Rightarrow y = 5 - \frac{1}{2}x.$$

By the Pythagorean Theorem, the height of such a triangle is $h = \sqrt{y^2 - \frac{1}{4}x^2}$, and so the area to be maximized is

$$A = \frac{1}{2}x\sqrt{y^2 - \frac{1}{4}x^2} \Rightarrow A(x) = \frac{1}{2}x\sqrt{(5 - \frac{1}{2}x)^2 - \frac{1}{4}x^2} = \frac{1}{2}x\sqrt{25 - 5x}, \quad 0 \le x \le 5.$$

(The upper limit of 5 is clear both because a value of $x$ larger than this would put a negative number under the radical, and because the single side cannot be more than half the perimeter.) In order to maximize $A(x)$, we compute

$$A'(x) = \frac{\frac{25}{2} - \frac{15}{4}x}{\sqrt{25 - 5x}}.$$

The critical values are $x = \frac{10}{3}, 5$, where we observe that $x = 5$ is also a boundary value. Checking $A(x)$ at the critical and boundary values, we find

$$\begin{aligned} A(0) &= 0 \\ A(\frac{10}{3}) &= \frac{5}{3}\sqrt{\frac{25}{3}} = \frac{25}{3\sqrt{3}} \\ A(5) &= 0. \end{aligned}$$

We conclude that the maximum area is $\frac{25}{3\sqrt{3}}$ and the side-lengths are $x = \frac{10}{3}$ and $y = 5 - \frac{1}{2}(\frac{10}{3}) = \frac{10}{3}$. That is, an equilateral triangle.

8. The fixed points solve

$$a = \frac{3}{4}a + \frac{1}{a} \Rightarrow \frac{1}{4}a = \frac{1}{a} \Rightarrow a^2 = 4.$$

We conclude that the fixed points are $\pm 2$. In order to use cobwebbing, we must sketch a graph of the function

$$f(a) = \frac{3}{4}a + \frac{1}{a}.$$

First, setting

$$f'(a) = \frac{3}{4} - \frac{1}{a^2} = 0,$$

we find that the critical points are $a = \pm\frac{2}{\sqrt{3}}, 0$. The function is increasing on $(-\infty, -\frac{2}{\sqrt{3}}] \cup [\frac{2}{\sqrt{3}}, \infty)$ and decreasing on $[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}]$. Next,

$$f''(a) = \frac{2}{a^3},$$

and so the only possible point of inflection is $a = 0$. The function is concave down on $(-\infty, 0)$

and concave up on $(0, \infty)$. Finally,

$$\begin{aligned}
\lim_{a\to-\infty} (\frac{3}{4}a + \frac{1}{a}) &= -\infty \\
f(-\frac{2}{\sqrt{3}}) &= -\sqrt{3} \\
\lim_{x\to 0^-} (\frac{3}{4}a + \frac{1}{a}) &= -\infty \\
\lim_{x\to 0^+} (\frac{3}{4}a + \frac{1}{a}) &= +\infty \\
f(\frac{2}{\sqrt{3}}) &= \sqrt{3} \\
\lim_{a\to-\infty} (\frac{3}{4}a + \frac{1}{a}) &= \infty
\end{aligned}$$

The plot of this function and the cobwebbing are depicted below. We conclude

$$\lim_{n\to\infty} a_n = 2.$$

9. In order to find the fixed points, we solve

$$x = 1 + \frac{2}{x},$$

which becomes (upon multiplication by $x$)

$$x^2 - x - 2 = (x-2)(x+1) = 0,$$

and the fixed points are $x = -1, 2$. In order to check for stability we set $f(x) = 1 + \frac{2}{x}$, and compute

$$f'(x) = -\frac{2}{x^2}.$$

We have

$$f'(-1) = -2 \Rightarrow -1 \text{ is unstable}$$
$$f'(2) = -\frac{1}{2} \Rightarrow 2 \text{ is stable.}$$

10. Since no value for $f(x)$ is given at either $x = 0$ or at $x = 1$, we cannot take a Riemann sum with left or right endpoints. We see, however, that the values of $x$ are precisely the midpoints of the subintervals in the partition $P = [0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1]$. The most reasonable Riemann sum is

$$\sum_{k=1}^{4} f(c_k)\Delta x_k,$$

where the $c_k$ are the interval midpoints. That is,

$$\sum_{k=1}^{4} f(c_k)\Delta x_k = (\frac{1}{2} + \frac{1}{3} - 1 - 2)\frac{1}{4} = -\frac{13}{24}.$$

11. In this case $\triangle x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$, and we use right endpoints $x_k = 1 + k\triangle x$. We have

$$\begin{aligned} A_n &= \sum_{k=1}^{n}[(1 + k\Delta x) + (1 + k\Delta x)^2]\Delta x \\ &= \sum_{k=1}^{n}[(1 + \frac{k}{n}) + (1 + 2\frac{k}{n} + \frac{k^2}{n^2})]\frac{1}{n} \\ &= \Big[\frac{1}{n}\sum_{k=1}^{n} 1 + \frac{1}{n^2}\sum_{k=1}^{n} k + \frac{1}{n}\sum_{k=1}^{n} 1 + \frac{2}{n^2}\sum_{k=1}^{n} k + \frac{1}{n^3}\sum_{k=1}^{n} k^2\Big] \\ &= \Big[1 + \frac{1}{n^2}\frac{n(n+1)}{2} + 1 + \frac{2}{n^2}\frac{n(n+1)}{2} + \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\Big]. \end{aligned}$$

Finally,

$$\lim_{n\to\infty} A_n = 1 + \frac{1}{2} + 1 + 1 + \frac{1}{3} = \frac{23}{6}.$$

12a. Using the substitution $u = e^x$, for which $du = e^x dx$, we find

$$\int \cos u du = \sin u + C = \sin(e^x) + C.$$

12b. We make the substitution $u = 1 + x$, with $du = dx$, and obtain

$$\begin{aligned} \int \frac{x}{\sqrt{u}}du &= \int \frac{u-1}{\sqrt{u}}du = \int u^{1/2} - u^{-1/2}du \\ &= \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + C = \frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C \\ &= (1+x)^{1/2}(\frac{2}{3}x - \frac{4}{3}) + C. \end{aligned}$$

12c. In this case, integrate by parts with $u = \cos^{-1} x$ and $dv = dx$, for which we have $du = -\frac{1}{\sqrt{1-x^2}} dx$ and $v = x$. The integral becomes

$$x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} dx.$$

For the remaining integral, we use fast substitution (since $u$ has already been used) to obtain

$$x \cos^{-1} x - \sqrt{1-x^2} + C.$$

13a. We make the substitution $u = 1 + x^3$ (or alternatively use fast substitution), so that $du = 3x^2 dx$, and the integral becomes

$$\int_2^{28} \frac{x^2}{\sqrt{u}} \frac{du}{3x^2} = \frac{1}{3} \int_2^{28} u^{-1/2} du = \frac{1}{3} \frac{u^{1/2}}{1/2} \Big|_2^{28} = \frac{2}{3} [\sqrt{28} - \sqrt{2}].$$

13b. We integrate by parts, setting

$$\begin{aligned} u &= x & dv &= \sec^2 x dx \\ du &= dx & v &= \tan x. \end{aligned}$$

We obtain

$$\begin{aligned} \int_0^{\frac{\pi}{4}} x \sec^2 x dx &= x \tan x \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x dx \\ &= \frac{\pi}{4} + \ln|\cos x| \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \ln(\frac{\sqrt{2}}{2}). \end{aligned}$$

14. First, we locate the points of intersection by solving

$$x^4 = 20 - x^2 \Rightarrow x^4 + x^2 - 20 = 0.$$

In general, fourth order equations are difficult to solve algebraically, but this is really a second order equation in the variable $x^2$, and it factors as

$$(x^2 - 4)(x^2 + 5) = 0,$$

so that the real roots are $x = \pm 2$. We observe that the upper graph is always $y = 20 - x^2$, and also take advantage of symmetry to compute the area as

$$A = 2 \int_0^2 (20 - x^2) - x^4 dx = 2[20x - \frac{x^3}{3} - \frac{x^5}{5}]_0^2 = 2[40 - \frac{8}{3} - \frac{32}{5}] = \frac{928}{15}.$$

15. We observe that the graph of $y = e^x$ is always above the graph of $y = e^{-x}$ on $[0, 2]$, and so according to the method of washers,

$$\begin{aligned} V &= \pi \int_0^2 (e^x)^2 - (e^{-x})^2 dx = \pi \int_0^2 e^{2x} - e^{-2x} dx \\ &= \frac{\pi}{2} [e^{2x} + e^{-2x}] \Big|_0^2 = \frac{\pi}{2} [e^4 + e^{-4} - 2], \end{aligned}$$

where in this case we used fast substitution.

16. First, the base of the triangle extends from $y = x^2$ up to $y = x$, so its length is $x - x^2$. The angle opposite the base is a right angle, so if we drop a line perpendicular to the base we divide the triangle into two 45-45-90 triangles. In this way, we see that the height of the triangle is $\frac{x-x^2}{2}$. (Alternatively, observe that the halves of the triangle can be rearranged into a square with sidelength $\frac{x-x^2}{2}$.) The area is

$$A(x) = \frac{1}{2}bh = \frac{1}{2}(x - x^2)\frac{x - x^2}{2} = \frac{1}{4}(x - x^2)^2.$$

The volume is

$$V = \frac{1}{4}\int_0^1 (x - x^2)^2 dx = \frac{1}{4}\int_0^1 x^2 - 2x^3 + x^4 dx = \frac{1}{4}[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}]_0^1 = \frac{1}{120}.$$