M611 Fall 2019 Assignment 10, due Monday Nov. 18

1. [10 pts] Compute the volume for a heat ball

$$E(\vec{x}, t; r) = \{(\vec{y}, s) : s \leq t, \Phi(\vec{x} - \vec{y}, t - s) \geq \frac{1}{r^n}\}.$$

2. [10 pts] Let $U \subset \mathbb{R}^n$ be open and bounded, $T > 0$, and assume $u \in C_1^2(U_T)$, where $U_T = U \times (0, T]$. Let

$$U_\epsilon = \{\vec{x} \in U : \text{dist } (\vec{x}, \partial U) > \epsilon\},$$

and for $\vec{x} \in U_\epsilon$ set

$$u^\epsilon(\vec{x}, t) = \eta_\epsilon * u(\cdot, t) = \int_{B(\vec{x},\epsilon)} \eta_\epsilon(\vec{x} - \vec{y}) u(\vec{y}, t) d\vec{y}.$$

a. Show that if u satisfies the heat equation on U_T then u^ϵ satisfies the heat equation on $U_T^\epsilon := U_\epsilon \times (0, T]$.

b. Show that the mixed partial derivatives $u^\epsilon_{tx_i}$ and $u^\epsilon_{x_it}$ both exist in U_T^ϵ and $u^\epsilon_{tx_i} = u^\epsilon_{x_it}$ in U_T^ϵ.

c. Use (a) and (b) to complete our proof from class of Theorem 2.3.3 in Evans.

3. [10 pts] (**Evans 2.5.17.**) We say $v \in C_1^2(U_T)$ is a *subsolution* of the heat equation if

$$v_t - \Delta v \leq 0 \quad \text{in } U_T.$$

a. Prove for a subsolution v that

$$v(\vec{x}, t) \leq \frac{1}{4r^n} \int\int_{E(\vec{x},t,r)} v(\vec{y}, s) \frac{|\vec{x} - \vec{y}|^2}{(t - s)^2} d\vec{y} ds$$

for all $E(\vec{x}, t; r) \subset U_T$.

b. Prove that therefore $\max_{\bar{U}_T} v = \max_{\Gamma_T} v$.

c. Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u solves the heat equation and $v := \phi(u)$. Prove v is a subsolution.

d. Prove $v := |Du|^2 + u_t^2$ is a subsolution, whenever u solves the heat equation.

4. [10 pts] (**Evans 5.10.5.**) Let U and V be open sets, with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta \equiv 1$ on V, $\zeta = 0$ near ∂U. (Hint: Take $V \subset\subset W \subset\subset U$ and mollify χ_W).

Note. Also check that $0 \leq \zeta \leq 1$.