M611 Fall 2019 Assignment 2, due Friday Sept. 13

1. [10 pts] Solve the ODE system

$$\begin{aligned}\frac{dy_1}{dt} &= y_2; \quad y_1(0) = 1\\ \frac{dy_2}{dt} &= 4y_1 + 3y_2 - 4y_3; \quad y_2(0) = 0\\ \frac{dy_3}{dt} &= y_1 + 2y_2 - y_3; \quad y_3(0) = 0.\end{aligned}$$

2. [10 pts] (**Jordan Canonical Form**) Although some matrices cannot be diagonalized, all matrices can be put into Jordan Canonical form. If a matrix $A \in \mathbb{C}^{n\times n}$ has k distinct eigenvalues $\{\lambda_i\}_{i=1}^k$ with respective *geometric* multiplicies $\{m_i\}_{i=1}^k$, then the Jordan canonical form of A is

$$J = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_m \end{pmatrix},$$

where

$$m = \sum_{i=1}^k m_i,$$

and each submatrix J_j has the form

$$J_j = \begin{pmatrix} \lambda_j & 1 & 0 & 0 \\ 0 & \lambda_j & 1 & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda_j \end{pmatrix}, \quad j = 1, 2, \ldots, m.$$

I.e., the number of times that λ_i appears in J is its algebraic multiplcity, while the number of Jordan blocks associated with λ_i is its geometric multiplicity. In this problem, we will see how to put a matrix into Jordan form, and how to use this form to solve the linear constant coefficient ODE specified by that matrix.

First, for an ODE

$$\frac{d\vec{y}}{dt} = A\vec{y},$$

we can make the change of variables $\vec{y} = P\vec{x}$, which leads to the new equation

$$\frac{d\vec{x}}{dt} = P^{-1}AP\vec{x}.$$

We would like to construct P in such a way that $P^{-1}AP$ is in Jordan canonical form. We can do this as follows: let its first n_1 columns comprise the generalized eigenvectors associated with λ_1, its next n_2 columns the generalized eigenvectors associated with λ_2 etc. Moreover, order these eigenvectors as follows: Let $\vec{v}_1$ satisfy

$$(A - \lambda_1 I)\vec{v}_1 = 0,$$

and then let $\vec{v}_k$ satisfy

$$(A - \lambda_1 I)\vec{v}_k = \vec{v}_{k-1},$$

continuing in this way until no such $\vec{v}_k$ exists. (The result is a Jordan chain, and the existing $\{\vec{v}_k\}_{k=1}^{p_1}$ will serve as the first p_1 columns of P. Also the first Jordan block of J will be of size $p_1 \times p_1$.) If λ_1 has geometric multiplicity greater than 1, take a second regular eigenvector and repeat the procedure above to get p_2 generalized eigenvectors. Continue in this way until the linearly independent regular eigenvectors associated with λ_1 have all been used, and then go to λ_2. Do exactly the same thing with λ_2, and continue until you run out of eigenvalues. With the resulting matrix P, the Jordan form of A is easily computed as

$$J = P^{-1}AP.$$

Here's the actual problem. Construct the Jordan form for

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix},$$

and use your result to solve the ODE

$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad \vec{y}_0 = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

3. [10 pts] Define a map on square matrices by setting

$$\rho(A, B) := \text{rank } (A - B),$$

for any square matrices $A, B \in \mathbb{C}^{n\times n}$. Show that ρ defines a metric.

4. [10 pts] Show that the sequence of functions $\{f_k\}_{k=1}^{\infty} \subset C([-1, 1])$ defined by

$$f_k(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ kx & 0 \leq x \leq \frac{1}{k} \\ 1 & \frac{1}{k} < x \leq 1 \end{cases}$$

is Cauchy in the metric

$$\rho(f, g) = \int_{-1}^{1} |f(x) - g(x)| dx,$$

but that it does not converge to a function in $C([-1, 1])$. (I.e., we are showing that the metric space $(C([-1, 1]), \rho)$ is not complete.)