M611 Fall 2019 Assignment 9, due Friday Nov. 8

1. [10 pts] (**Evans 2.5.12.**) Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.
(i) Show $u_\lambda(\vec{x}, t) := u(\lambda\vec{x}, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.
(ii) Use (i) to show $v(\vec{x}, t) := \vec{x} \cdot Du(\vec{x}, t) + 2tu_t(\vec{x}, t)$ solves the heat equation as well.

2. [10 pts] Suppose $f \in C_1^2(\mathbb{R}^n \times \mathbb{R})$ has compact support and set

$$u(\vec{x}, t) = \int_{-\infty}^{+\infty} \int_{\mathbb{R}^n} \Phi(\vec{x} - \vec{y}, t - s) f(\vec{y}, s) d\vec{y} ds.$$

Show that $u \in C_1^2((\mathbb{R}^n \times \mathbb{R}) \backslash (\mathbb{R}^n \times \{0\}))$, and

$$u_t - \Delta u = f; \quad \text{in } (\mathbb{R}^n \times \mathbb{R}) \backslash (\mathbb{R}^n \times \{0\}).$$

3. [10 pts] (**Evans 2.5.15.**) Given $g : [0, \infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x, t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds$$

for a solution of the initial/boundary-value problem

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in } \mathbb{R}_+ \times (0, \infty) \\ u &= 0 \quad \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u &= g \quad \text{on } \{x = 0\} \times [0, \infty). \end{aligned}$$

(Hint: Let $v(x, t) := u(x, t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.)

Note. Although we assumed $f \in C_1^2(\mathbb{R}^n \times [0, \infty))$ with compact support in our proof of Theorem 2.3.2, equation (13) is valid much more generally, and Evans (presumably) means for you to apply it formally here.

4. [10 pts] Suppose $g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and set

$$u(\vec{x}, t) = \int_{\mathbb{R}^n} \Phi(\vec{x} - \vec{y}, t) g(\vec{y}) d\vec{y},$$

where Φ denotes the fundamental solution for the heat equation. Show that for all $t > 0$:

a. $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq \|g\|_{L^\infty(\mathbb{R}^n)}$
b. $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-n/2}\|g\|_{L^1(\mathbb{R}^n)}$.
c. $\|u(\cdot, t)\|_{L^1(\mathbb{R}^n)} \leq \|g\|_{L^1(\mathbb{R}^n)}$.
d. $\int_{\mathbb{R}^n} u(\vec{x}, t) d\vec{x} = \int_{\mathbb{R}^n} g(\vec{x}) d\vec{x}$.
e. $\|u_{x_i}(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-1/2}\|g\|_{L^\infty(\mathbb{R}^n)}$.
f. $\|u_{x_i}(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n+1}{2}}\|g\|_{L^1(\mathbb{R}^n)}$.