M612 Spring 2020, Assignment 10, due Friday, Apr. 24

1. [10 pts] (**Evans 6.6.6.**) Suppose U is connected and ∂U consists of two disjoint, closed sets Γ_1 and Γ_2. Define what it means for u to be a weak solution of Poisson's equation with *mixed Dirichlet-Neumann boundary conditions*:

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_2. \end{aligned}$$

Discuss the existence and uniqueness of weak solutions.

Note. Similarly as with with Problem 6.6.5, by "discuss" Evans presumably means rigorously determine conditions on f under which we are guaranteed the existence of weak solutions.

2. [10 pts] In Step 7 of the proof of Theorem 6.3.1, verify the inequality

$$\int_U \zeta^2 |Du|^2 d\vec{x} \le C \int_U |f|^2 + |u|^2 d\vec{x},$$

arising from substitution of $v = \zeta^2 u$ into

$$\sum_{i,j=1}^{n} \int_U a^{ij} u_{x_i} v_{x_j} d\vec{x} = \int_U \tilde{f} v d\vec{x}.$$

3. [10 pts] (**Evans 6.6.7.**) Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth, with $c(0) = 0$ and $c' \ge 0$. Prove $u \in H^2(\mathbb{R}^n)$.
(Hint: Mimic the proof of Theorem 1 in Section 6.3.1, but without the cutoff function ζ.)

Note. Assume that for u as described in the problem statement, $c(u) \in L^2(\mathbb{R}^n)$. You should be able to get an estimate of the form

$$\|D^2 u\|_{L^2(\mathbb{R}^n)} \le C \|f\|_{L^2(\mathbb{R}^n)}.$$

4. [10 pts] In Step 7 of the proof of Theorem 6.3.4, show that $u' \in H^1(U')$ and

$$u' = 0 \quad \text{on } \partial U' \cap \{y_n = 0\}$$

in the trace sense.