M612 Spring 2020 Assignment 4, due Fri., Feb. 14

1. [10 pts] Suppose $U \subset \mathbb{R}^n$ is open and $u, v \in H_0^1(U)(= W_0^{1,2}(U))$. Prove that we can integrate by parts:

$$\int_U u v_{x_i} d\vec{x} = -\int_U u_{x_i} v d\vec{x}.$$

2. [10 pts] (**Evans 5.10.4.**) Assume $n = 1$ and $u \in W^{1,p}(0,1)$ for some $1 \leq p < \infty$.

(a) Show that u is equal a.e. to an absolutely continuous function, and u' (which exists a.e.) belongs to $L^p(0,1)$.

(b) Prove that if $1 < p < \infty$, then

$$|u(x) - u(y)| \leq |x - y|^{1-\frac{1}{p}} \Big(\int_0^1 |u'|^p dt \Big)^{1/p},$$

for a.e. $x, y \in [0,1]$.

Note. We say u is absolutely continuous on $[0,1]$, often denoted $u \in AC[0,1]$, provided that for every $\epsilon > 0$ there exists $\delta > 0$ so that for any finite collection of interval subsets of $[0,1]$, $\{[a_j, b_j]\}_{j=1}^N$, mutually disjoint, except possibly at the endpoints, satisfying $\sum_{j=1}^N (b_j - a_j) \leq \delta$ we have $\sum_{j=1}^N |u(b_j) - u(a_j)| \leq \epsilon$.

3. [10 pts] Show that if $u \in H^1(\mathbb{R})$, and u' denotes the weak derivative of u, then

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h},$$

where the limit is in the $L^2(\mathbb{R})$ sense.

4. [10 pts] Fill in a step in the proof of Theorem 5.3.3 by verifying that (in the notation of that proof)

$$\|D^\alpha v^\epsilon - D^\alpha u_\epsilon\|_{L^p(V)} \to 0$$

as $\epsilon \to 0$.

Note. The key, of course, is to deal with the appearance of ϵ both in the mollifier and in the function being mollifed. Keep in mind that we cannot specify

$$u^\epsilon(\vec{x}) = \eta_\epsilon * u(\vec{x})$$

for $\vec{x} \in V$ because for any $\epsilon > 0$ there would be $\vec{x} \in V$ so that $B(\vec{x}, \epsilon)$ is not in U, where u is defined. One approach is to note that v^ϵ and u_ϵ are both defined on some W_ϵ so that $V \subset\subset W_\epsilon$ and proceed as in the proof of Theorem A.C.7, starting with Step 4.