M612 Spring 2020, Assignment 9, due Fri., Apr. 17

1. [10 pts] Suppose H is a real Hilbert space, and $A : H \to H$ is a bounded linear operator. Show the following.

a. The adjoint operator A^* is the unique operator $A^* : H \to H$ so that

$$(Au, v) = (u, A^*v), \quad \forall\, u, v \in H.$$

b. $A^* : H \to H$ is a bounded linear operator, with $\|A^*\| = \|A\|$.

2. [10 pts] (**Evans 6.6.2.**) Let

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu.$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided

$$c(\vec{x}) \geq -\mu \quad (\vec{x} \in U).$$

3. [10 pts] (**Evans 6.6.3.**) A function $u \in H_0^2(U)$ is a weak solution of this boundary-value problem for the *biharmonic equation*

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } U \\ u = \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial U \end{aligned} \tag{*}$$

provided

$$\int_U \Delta u \Delta v d\vec{x} = \int_U f v d\vec{x}$$

for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove that there exists a unique weak solution of (*).

4. [10 pts] (**Evans 6.6.4.**) Assume U is connected. A function $u \in H^1(U)$ is a weak solution of Neumann's problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial U \end{aligned} \tag{*}$$

if

$$\int_U Du \cdot Dv d\vec{x} = \int_U f v d\vec{x}$$

for all $v \in H^1(U)$. Suppose $f \in L^2(U)$. Prove (*) has a weak solution if and only if

$$\int_U f d\vec{x} = 0.$$

5. [10 pts] Suppose $U \subset \mathbb{R}^n$ is open and bounded, $a^{ij}, b^i, c \in L^\infty(U)$, and

$$Lu := -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu$$

is uniformly elliptic with constant $\theta > 0$. Let C_p denote the Poincare constant so that

$$\|u\|_{L^2(U)} \leq C_p \|Du\|_{L^2(U)},$$

for all $u \in H_0^1(U)$ and suppose

$$C_p \sum \|b^i\|_{L^\infty(U)} + C_p^2 \|c_-\|_{L^\infty(U)} < \theta,$$

where $c_-(\vec{x}) := \min(c(\vec{x}), 0)$. Show that there exists a unique weak solution $u \in H_0^1(U)$ to the PDE

$$\begin{aligned} Lu &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U, \end{aligned}$$

for $f \in H^{-1}(U)$, and that there exists a constant C so that

$$\|u\|_{H^1(U)} \leq C \sum_{i=0}^{n} \|f^i\|_{L^2(U)},$$

where $\{f^i\}_{i=0}^n \subset L^2(U)$ is a choice of functions so that

$$f = f^0 - \sum_{i=1}^{n} f^i_{x_i}.$$

6. [10 pts] (**Evans 6.6.5.**) Explain how to define $u \in H^1(U)$ to be a weak solution of Poisson's equation with *Robin boundary conditions*:

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u + \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial U. \end{aligned}$$

Discuss the existence and uniqueness of a weak solution for a given $f \in L^2(U)$.

Note. By "discuss" Evans presumably means determine conclusively whether or not we are guaranteed the existence of weak solutions for such f. The standard approach to coercivity for problems like this is to argue by contradiction.