Second Order Elliptic PDE: Weak Solutions

MATH 612, Texas A&M University

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Reference Material

Most texts on second order elliptic PDE are indebted to one extent or another to the standard reference, "Elliptic partial differential equations of second order," by David Gilbarg and Neil Trudinger. This is true of Evans's book, and it's also a resource I'll use for some of our material.

Second Order Elliptic PDE

For $U \subset \mathbb{R}^n$ open and bounded, we'll be interested in PDE of the form

$$\begin{aligned} Lu &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U, \end{aligned}$$

where L has either the divergence form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu,$$

or the non-divergence form

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu.$$

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Notes. 1. Our standard example from last semester was $Lu = -\Delta u$, for which $a^{ij} = \delta_{ij}$ (the Kronecker delta function), $b^i = 0$ for all $i \in \{1, 2, \dots n\}$, $c = 0$.

2. If $a^{ij} \in C^1(U)$, it's easy to convert from one form to the other. For example, if L is originally in non-divergence form, we can write

$$
\begin{aligned}
Lu &= -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu \\
&= -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i,j=1}^{n} a^{ij}_{x_j} u_{x_i} + \sum_{i=1}^{n} b^i u_{x_i} + cu \\
&= -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} \Big(b^i + \sum_{j=1}^{n} a^{ij}_{x_j} \Big) u_{x_i} + cu,
\end{aligned}
$$

which is in divergence form with b^i replaced by $\tilde{b}^i = \Big(b^i + \sum_{j=1}^{n} a^{ij}_{x_j} \Big)$. (Converting in the other direction is similar.)

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3. We'll find that the divergence form is better suited for developing energy estimates, while non-divergence form is better suited for maximum principle techniques.

4. As noted in our discussion of PDE classification last semester, if $u \in C^2(U)$, we can take $a^{ij} = a^{ji}$ without loss of generality. In the current setting, we'll take this symmetry as an assumption. Recall that in this setting if we let A denote the matrix with components (a^{ij}), then L is called elliptic at a point $\vec{x} \in U$ provided the eigenvalues of $A(\vec{x})$ all have the same sign.

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5. If we use the notation $A = (a^{ij})$ and $\vec{b} = (b^1\ b^2\ \dots\ b^n)$, then we can express the divergence and non-divergence forms of L in the index-free forms

$$\begin{aligned} Lu &= -\nabla \cdot ((Du)A) + \vec{b} \cdot Du + cu \quad \text{(Divergence form)} \\ Lu &= -A : D^2 u + \vec{b} \cdot Du + cu \quad \text{(Non-divergence form).} \end{aligned}$$

Here,

$$A : B := \sum_{i,j=1}^{n} a^{ij} b^{ij}.$$

Uniform Ellipticity

Definition. We say that L (either form) is uniformly elliptic in U if there exists a constant $\theta > 0$ so that

$$\sum_{i,j=1}^{n} a^{ij}(\vec{x})\xi_i\xi_j \geq \theta|\vec{\xi}|^2$$

for a.e. $\vec{x} \in U$ and all $\vec{\xi} \in \mathbb{R}^n$. I.e.,

$$\vec{\xi}^T A\vec{\xi} \geq \theta|\vec{\xi}|^2.$$

Notes. 1. For the Laplacian, $A = I$, so $\vec{\xi}^T A\vec{\xi} = |\vec{\xi}|^2$, and we see that $\theta = 1$.

2. More generally, according to the min-max principle for matrices, this condition asserts that for a.e. $\vec{x} \in U$ the eigenvalues of $A(\vec{x})$ are all bounded below by θ. In this way, uniform ellipticity implies ellipticity.

Weak Solutions

First, suppose u is a classical solution of the divergence-form problem

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu = f. \qquad (*)$$

If we multiply this equation by $v \in C_c^\infty(U)$ and integrate the first sum by parts, we obtain the relation

$$\int_U \left\{ \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \right\} d\vec{x} = \int_U fv d\vec{x}.$$

This motivates our week formulation of (*). For this, we will assume a^{ij}, b^i, $c \in L^\infty(U)$ for all $i, j \in \{1, 2, \ldots, n\}$, and $f \in L^2(U)$.

Weak Solutions

Definitions.

(i) We specify the bilinear form

$$B[u,v] := \int_U \left\{ \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \right\} d\vec{x}.$$

(ii) We say that $u \in H_0^1(U)$ is a weak solution of the divergence form equation

$$\begin{aligned} Lu &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U, \end{aligned}$$

if $B[u,v] = (f,v)$ for all $v \in H_0^1(U)$. Here, $(\cdot,\cdot)$ denotes $L^2(U)$ inner product.

Weak Solutions

Notes. 1. Clearly, any classical solution is a weak solution (assuming the PDE makes sense classically).

2. It's customary to continue referring to the strong formulation of a PDE, even when discussing weak solutions, and that's often what we'll do. I.e., we'll write things like $u \in H_0^1(U)$ satisfies

$$\Delta u = f,$$

even though u may not have weak derivatives to second order.

Weak Solutions

3. More generally, if $f \in H^{-1}(U)$ and $u \in H_0^1(U)$ satisfies

$$B[u, v] = \langle f, v \rangle$$

for all $v \in H_0^1(U)$, we'll write

$$\begin{aligned} Lu &= f^0 - \sum_{i=1}^{n} f^i_{x_i} \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U, \end{aligned}$$

where we recall $\langle f, v \rangle = \int_U f^0 v + \sum_{i=1}^{n} f^i v_{x_i} d\vec{x}$.

Weak Solutions

4. Suppose $f \in L^2(U)$, $g \in L^2(\partial U)$, and ∂U is C^1. We say that $u \in H^1(U)$ is a weak solution of

$$\begin{aligned} Lu &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U, \end{aligned}$$

if

$$B[u, v] = (f, v)$$

for all $v \in H_0^1(U)$ and u is equal to g on ∂U in the trace sense: $Tu = g$, where T is the trace operator from Theorem 5.5.1. Similarly as in Note 3, we can extend this to $f \in H^{-1}(U)$.

Weak Solutions

5. Suppose that in Note 4 we can find $w \in H^1(U)$ so that $Tw = g$. We can set $\tilde{u} = u - w$ so that $\tilde{u}$ is a weak solution of

$$
\begin{aligned}
L\tilde{u} &= f - Lw \quad \text{in } U \\
\tilde{u} &= 0 \quad \text{on } \partial U.
\end{aligned}
$$

Here, Lw doesn't make sense in the strong formulation, but we mean $Lw \in H^{-1}(U)$ in the sense that

$$
\langle Lw, v \rangle = B[w, v] \quad \forall\, v \in H_0^1(U).
$$

I.e., in the weak formulation we have

$$
\begin{aligned}
B[\tilde{u}, v] = B[u - w, v] &= B[u, v] - B[w, v] \\
&= (f, v) - B[w, v].
\end{aligned}
$$