Second Order Elliptic PDE: The Lax-Milgram Theorem

MATH 612, Texas A&M University

Spring 2020

The Lax-Milgram Theorem

Let H denote a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\| = (\cdot, \cdot)^{1/2}$. We'll continue to use $\langle \cdot, \cdot \rangle$ to denote the action of an element of H^* on an element of H.

Theorem 6.2.1. (Lax-Milgram Theorem) Suppose $B : H \times H \to \mathbb{R}$ is a bilinear form for which there exist constants $\alpha, \beta > 0$ so that

$$|B[u, v]| \leq \alpha \|u\| \|v\|, \quad \forall\, u, v \in H \quad \text{(boundedness)}$$
$$B[u, u] \geq \beta \|u\|^2, \quad \forall\, u \in H \quad \text{(coercivity)}.$$

Then for each $f \in H^*$, there exists a unique $u \in H$ so that

$$B[u, v] = \langle f, v \rangle$$

for all $v \in H$.

The Lax-Milgram Theorem

Notes. 1. If B is symmetric (i.e., $B[u, v] = B[v, u]$ for all $u, v \in H$), then $B[u, v]$ is an inner product on H, and this is just the Riesz Representation Theorem.

2. A similar statement is true for a complex Hilbert space H, assuming B is sesquilinear. There are also many extensions.

3. Before proving the theorem, we'll work through a simple application.

Application to Poisson's Equation

For $U \subset \mathbb{R}^n$ open and bounded, consider Poisson's equation,

$$-\Delta u = f \in H^{-1}(U), \quad \text{in } U$$
$$u = 0, \quad \text{on } \partial U.$$

The weak formulation for this problem is

$$\int_U \sum_{i,j=1}^{n} \delta_{ij} u_{x_i} v_{x_j} d\vec{x} = \langle f, v \rangle, \quad \forall\, v \in H_0^1(U).$$

If we define the bilinear form

$$B[u,v] = \int_U \sum_{i,j=1}^{n} \delta_{ij} u_{x_i} v_{x_j} d\vec{x} = \int_U Du \cdot Dv d\vec{x},$$

then we can express this weak formulation as

$$B[u,v] = \langle f, v \rangle, \quad \forall\, v \in H_0^1(U).$$

Application to Poisson's Equation

We see that our Hilbert space for the Lax-Milgram Theorem is $H_0^1(U)$, and in order to apply the theorem, we only need to check that $B[u, v]$ is bounded and coercive.

For boundedness, we have

$$\begin{aligned}|B[u,v]| &= \left| \int_U Du \cdot Dv d\vec{x} \right| \\ &\overset{\text{c.s.}}{\leq} \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} \leq \|u\|_{H^1(U)} \|v\|_{H^1(U)}.\end{aligned}$$

I.e., $\alpha = 1$.

Application to Poisson's Equation

For coercivity, we start with

$$B[u,u] = \int_U |Du|^2 d\vec{x} = \|Du\|^2_{L^2(U)}.$$

We recall that Poincare's inequality (from Theorem 5.6.3) asserts that there exists a constant C, depending only on n and U, so that

$$\|u\|_{L^2(U)} \leq C\|Du\|_{L^2(U)},$$

for all $u \in H^1_0(U)$.

Application to Poisson's Equation

We can write

$$\begin{aligned} B[u,u] &= \frac{1}{2}\|Du\|^2_{L^2(U)} + \frac{1}{2}\|Du\|^2_{L^2(U)} \\ &\geq \frac{1}{2C^2}\|u\|^2_{L^2(U)} + \frac{1}{2}\|Du\|^2_{L^2(U)} \\ &\geq \beta\left(\|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)}\right) \\ &= \beta\|u\|^2_{H^1(U)}, \end{aligned}$$

for all $u \in H^1_0(U)$. (Here, $\beta = \min\{\frac{1}{2C^2}, \frac{1}{2}\}$.)

The Lax-Milgram Theorem allows us to conclude immediately that there exists a unique solution $u \in H^1_0(U)$ to Poisson's equation.

Proof of the Lax-Milgram Theorem

1. First, for each fixed $u \in H$, the mapping $T_u v = B[u,v]$ is a bounded linear functional on H. I.e.,

$$|T_u v| = |B[u,v]| \leq \alpha \|u\| \|v\|,$$

and linearity follows from the bilinearity of B.

We can conclude from the Riesz Representation Theorem that there exists a unique $w \in H$ so that

$$B[u,v] = (w,v), \quad \forall\, v \in H.$$

Let's denote by $A : H \to H$ the map that takes $u \in H$ as input and returns $w \in H$ in this way. I.e.,

$$B[u,v] = (Au, v), \quad \forall v \in H.$$

Proof of the Lax-Milgram Theorem

2. **Claim 1.** $A \in H^*$.

To see that A is linear, let $\lambda_1, \lambda_2 \in \mathbb{R}$, $u_1, u_2 \in H$, and compute

$$\begin{aligned}\Big(A(\lambda_1 u_1 + \lambda_2 u_2), v\Big) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \\ &= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) \\ &= (\lambda_1 Au_1 + \lambda_2 Au_2, v).\end{aligned}$$

Since this is true for all $v \in H$, we can conclude that

$$A(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 Au_1 + \lambda_2 Au_2.$$

Proof of the Lax-Milgram Theorem

To see that A is bounded, we first note that there's nothing to show if $Au = 0$. If $Au \neq 0$, we compute

$$\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha \|u\| \|Au\|.$$

Dividing by $\|Au\|$, we see that

$$\|Au\| \leq \alpha \|u\|.$$

Proof of the Lax-Milgram Theorem

3. **Claim 2.** A is injective, and the range of A, $R(A)$, is closed in H.

To see that A is injective, we use coercivity to write

$$\beta\|u\|^2 \leq B[u,u] = (Au, u) \overset{c.s.}{\leq} \|Au\|\|u\|.$$

If $\|u\| \neq 0$, we divide to see that

$$\|Au\| \geq \beta\|u\|.$$

(This is trivially true if $\|u\| = 0$.) In particular, if $u_1, u_2 \in H$, then $\|A(u_1 - u_2)\| \geq \beta\|u_1 - u_2\|$, from which it's clear that A is injective. (I.e., $u_1 \neq u_2 \implies Au_1 \neq Au_2$.)

Proof of the Lax-Milgram Theorem

To see that $R(A)$ is closed in H, let $\{u_j\}_{j=1}^{\infty} \subset H$ satisfy $Au_j \to w$ for some $w \in H$. We need to show that there exists $u \in H$ so that $Au = w$ (i.e., $w \in R(A)$).

For this, we notice that

$$\|u_i - u_j\| \leq \frac{1}{\beta}\|Au_i - Au_j\|.$$

The sequence $\{Au_j\}_{j=1}^{\infty}$ converges, so it must be Cauchy, so we see that $\{u_j\}_{j=1}^{\infty}$ must be Cauchy, and so must converge to some $u \in H$. Since A is bounded,

$$\|Au - w\| = \lim_{j\to\infty} \|Au - Au_j\| \leq \alpha \lim_{j\to\infty} \|u - u_j\| = 0.$$

I.e., $Au = w$. (Alternatively, since A is bounded, we know from the Closed Graph Theorem that A is closed, and this allows us to conclude $Au = w$.)

Proof of the Lax-Milgram Theorem

4. **Claim 3.** $R(A) = H$.

Suppose not, and recall that since $R(A)$ is closed, we can write

$$H = R(A) \oplus R(A)^{\perp}.$$

If $R(A) \subsetneq H$, then we can find $w \in R(A)^{\perp} \setminus \{0\}$, and for this w we will have

$$0 \neq \beta \|w\|^2 \leq B[w, w] = (Aw, w) = 0,$$

which is a contradiction.

We can conclude from the Bounded Inverse Theorem that A has a bounded linear inverse, A^{-1}.

Proof of the Lax-Milgram Theorem

5. According to the Riesz Representation Theorem, for each $f \in H^*$ we can find a unique $w \in H$ so that

$$\langle f, v \rangle = (w, v)$$

for all $v \in H$. In this way, we see that we can solve

$$B[u, v] = \langle f, v \rangle, \quad \forall v \in H$$

by solving

$$B[u, v] = (w, v), \quad \forall v \in H.$$

Recalling that

$$B[u, v] = (Au, v), \quad \forall v \in H,$$

we see that $Au = w$, and the solution we're looking for is $u = A^{-1} w$.

Proof of the Lax-Milgram Theorem

6. For uniqueness, suppose u and $\tilde{u}$ are two solutions so that

$$\begin{aligned} B[u, v] &= \langle f, v \rangle \quad \forall v \in H \\ B[\tilde{u}, v] &= \langle f, v \rangle \quad \forall v \in H. \end{aligned}$$

Subtracting and using linearity, we see that

$$B[u - \tilde{u}, v] = 0, \quad \forall v \in H.$$

Take $v = u - \tilde{u}$, and notice that (from coercivity)

$$\|u - \tilde{u}\|^2 \leq \frac{1}{\beta} B[u - \tilde{u}, u - \tilde{u}] = 0.$$

I.e., $u = \tilde{u}$. $\square$