Second Order Elliptic PDE: Energy Estimates

MATH 612, Texas A&M University

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Energy Estimates

For general second order (uniformly) elliptic operators L (in divergence form), we need to check conditions under which the assumptions of the Lax-Milgram Theorem hold on the bilinear form

$$B[u,v] = \int_U \Big\{ \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \Big\} d\vec{x}. \qquad (*)$$

Theorem 6.2.2. Suppose $U \subset \mathbb{R}^n$ is open and bounded, a^{ij}, b^i, $c \in L^\infty(U)$ ($\forall\, i,j \in \{1,2,\ldots n\}$), and L is uniformly elliptic. Then for $B[u,v]$ in (*) there exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$, so that:

(i) $|B[u,v]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)}$

(ii) $B[u,u] \geq \beta \|u\|^2_{H^1(U)} - \gamma \|u\|^2_{L^2(U)}$

for all $u, v \in H^1_0(U)$.

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Note. The inequality in (ii) is sometimes referred to as Gårding's inequality.

Proof of Theorem 6.2.2. 1. For (i), we can write

$$
\begin{aligned}
|B[u,v]| &\leq \int_U \Big\{ \sum_{i,j=1}^n |a^{ij}||u_{x_i}||v_{x_j}| + \sum_{i=1}^n |b^i||u_{x_i}||v| + |c||u||v| \Big\} d\vec{x} \\
&\overset{c.s.}{\leq} C_1 \Big(\|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} + \|Du\| \|v\|_{L^2(U)} + \|u\|_{L^2(U)} \|v\|_{L^2(U)} \Big) \\
&\leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)},
\end{aligned}
$$

where we can take $\alpha = 3C_1$.

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2. For (ii), using uniform ellipticity, we can write

$$\sum_{i,j=1}^{n} a^{ij} u_{x_i} u_{x_j} \geq \theta |Du|^2,$$

for some $\theta > 0$. This allows us to compute

$$\theta \int_U |Du|^2 d\vec{x} \leq \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} u_{x_j} d\vec{x} = B[u,u] - \int_U \sum_{i=1}^{n} b^i u_{x_i} u + cu^2 d\vec{x}$$

$$\leq B[u,u] + C_2 \int_U |Du||u| + |u|^2 d\vec{x}.$$

At this point, we'll use Young's inequality,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b \geq 0, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p, q < \infty.$$

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For any $\epsilon > 0$, we can write

$$\begin{aligned} ab &= (p\epsilon)^{1/p} a \frac{b}{(p\epsilon)^{1/p}} \leq \frac{p\epsilon a^p}{p} + \frac{b^q}{q(p\epsilon)^{q/p}} \\ &= \epsilon a^p + C_\epsilon b^q. \end{aligned}$$

This is sometimes referred to as the ϵ-Young's inequality or the Peter-Paul inequality. For us, $p = 2$, so we'll use $ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$ to write

$$|Du||u| \leq \epsilon |Du|^2 + \frac{1}{4\epsilon}|u|^2.$$

This allows us to write

$$\theta \int_U |Du|^2 d\vec{x} \leq B[u,u] + C_2 \int_U \epsilon |Du|^2 + (1 + \frac{1}{4\epsilon})|u|^2 d\vec{x}.$$

We choose $\epsilon > 0$ small enough so that $C_2 \epsilon \leq \frac{\theta}{2}$.

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This choice gets us to

$$\frac{\theta}{2}\int_U |Du|^2 d\vec{x} \leq B[u,u] + C_2(1+\frac{1}{4\epsilon})\int_U |u|^2 d\vec{x},$$

from which we see that

$$B[u,u] \geq \frac{\theta}{2}\|Du\|^2_{L^2(U)} - C_2(1+\frac{1}{4\epsilon})\|u\|^2_{L^2(U)}.$$

The claim now follows from Poincare's inequality,

$$\|u\|_{L^2(U)} \leq C_3\|Du\|_{L^2(U)}, \quad \forall u \in H^1_0(U),$$

applied precisely as in our analysis of Poisson's equation. (Alternatively, in this case, we could simply add and subtract $\frac{\theta}{2}\|u\|^2_{L^2(U)}$ to the right-hand side, and subsume the subtracted part into $C_2(1+\frac{1}{4\epsilon})\|u\|^2_{L^2(U)}$.) □

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The appearance of γ in Theorem 6.2.2 is clearly a problem for coercivity. Notice that for the operator

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu,$$

with $c(\vec{x}) \geq 0$ for a.e. $\vec{x} \in U$, Step 2 of our proof becomes

$$\theta \int_U |Du|^2 d\vec{x} \leq B[u,u] - c \int_U u^2 d\vec{x}.$$

In this case,

$$B[u,u] \geq \theta \int_U |Du|^2 d\vec{x},$$

and coercivity follows from Poincare's inequality.

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Theorem 6.2.3. Let the assumptions of Theorem 6.2.2 hold, and let γ be as in the statement of Theorem 6.2.2. Then for any $\mu \geq \gamma$ and any $f \in H^{-1}(U)$, there exists a unique weak solution $u \in H_0^1(U)$ to the divergence-form equation

$$\begin{aligned} Lu + \mu u &= f, \quad \text{in } U \\ u &= 0, \quad \text{on } \partial U. \end{aligned}$$

Proof. If we let $B[u, v]$ denote the bilinear form associated with L, and we let $B_\mu[u, v]$ denote the bilinear form associated with $L + \mu$, then

$$B_\mu[u, v] = B[u, v] + \mu(u, v).$$

Then:

$$|B_\mu[u, v]| \leq |B[u, v]| + \mu|(u, v)| \leq (\alpha + \mu)\|u\|_{H^1(U)}\|v\|_{H^1(U)}.$$

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Likewise,

$$\begin{aligned} B_\mu[u,u] &= B[u,u] + \mu\|u\|^2_{L^2(U)} \\ &\geq \beta\|u\|^2_{H^1(U)} - \gamma\|u\|^2_{L^2(U)} + \mu\|u\|^2_{L^2(U)} \\ &\geq \beta\|u\|^2_{H^1(U)}. \end{aligned}$$

The claim now follows from the Lax-Milgram Theorem. $\square$

Note. We will use the observation from this theorem that the operator $L_\mu := L + \mu I : H^1_0(U) \to H^{-1}(U)$ is a bijection (existence $\implies$ surjectivity, uniqueness $\implies$ injectivity). (Notice particularly that we're again using notation for the strong problem, but referring to the weak formulation).

Why "Energy Estimates"

The motivation for this terminology is arguably clearest in connection with the wave equation, so let's discuss it in that context. Consider

$$\begin{aligned} u_{tt} &= \Delta u; \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ u(\vec{x}, 0) &= g(\vec{x}); \quad \vec{x} \in \mathbb{R}^n \\ u_t(\vec{x}, 0) &= h(\vec{x}); \quad \vec{x} \in \mathbb{R}^n. \end{aligned}$$

As boundary conditions, we'll assume $u(\vec{x}, t) \to 0$ as $|\vec{x}| \to \infty$ in such a way that we don't have any boundary terms when we integrate by parts in $\vec{x}$.

Suppose $u(\vec{x}, t)$ solves this equation. We can multiply the equation by $u_t(\vec{x}, t)$ and integrate both sides on $\mathbb{R}^n \times [0, T]$ for some $T > 0$. We get

$$\int_0^T \int_{\mathbb{R}^n} u_{tt} u_t \, d\vec{x} dt = \int_0^T \int_{\mathbb{R}^n} \Delta u u_t \, d\vec{x} dt.$$

Why "Energy Estimates"

For the integral on the left, we can write

$$\int_0^T \int_{\mathbb{R}^n} u_{tt} u_t \, d\vec{x} dt = \int_0^T \int_{\mathbb{R}^n} \frac{d}{dt} \frac{u_t^2}{2} d\vec{x} dt$$
$$= \frac{1}{2} \int_{\mathbb{R}^n} u_t(\vec{x}, T)^2 - u_t(\vec{x}, 0)^2 d\vec{x}.$$

Likewise, for the integral on the right we can integrate by parts to see that

$$\int_0^T \int_{\mathbb{R}^n} \Delta u u_t d\vec{x} dt = -\int_0^T \int_{\mathbb{R}^n} Du \cdot Du_t d\vec{x} dt$$
$$= -\int_0^T \int_{\mathbb{R}^n} \frac{d}{dt} \frac{|Du|^2}{2} d\vec{x} dt$$
$$= -\frac{1}{2} \int_{\mathbb{R}^n} |Du(\vec{x}, T)|^2 - |Du(\vec{x}, 0)|^2 d\vec{x}.$$

Why "Energy Estimates"

If we equate these two expressions (and multiply by 2), we see that

$$\int_{\mathbb{R}^n} u_t(\vec{x}, T)^2 - u_t(\vec{x}, 0)^2 d\vec{x} = -\int_{\mathbb{R}^n} |Du(\vec{x}, T)|^2 - |Du(\vec{x}, 0)|^2 d\vec{x}.$$

Rearranging terms, we find

$$\int_{\mathbb{R}^n} u_t(\vec{x}, T)^2 + |Du(\vec{x}, T)|^2 d\vec{x} = \int_{\mathbb{R}^n} u_t(\vec{x}, 0)^2 + |Du(\vec{x}, 0)|^2 d\vec{x},$$

for all $T > 0$. I.e., the energy

$$E(T) = \int_{\mathbb{R}^n} u_t(\vec{x}, T)^2 + |Du(\vec{x}, T)|^2 d\vec{x}$$

is constant in T.

Here, recall that in our derivation of the wave equation last semester, u_t corresponded with velocity of a string, membrane, or elastic solid, and $|Du|$ served as a measure of stretching. In particular, $E(T)$ can be viewed as a sum of kinetic energy and potential energy.

Why "Energy Estimates"

Generally, arguments carried out in this way, by multiplying the equation by some quantity and integrating (often by parts), are called energy arguments. Inequalities obtained in this way are called energy estimates.

Recall that we obtained our form of $B[u, v]$ by multiplying our equation

$$-\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu = f,$$

by v and integrating by parts.