Second Order Elliptic PDE: The Resolvent Operator

MATH 612, Texas A&M University

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Spectral Theory for Compact Operators

Suppose X denotes a real Banach space and $A : X \to X$ denotes a bounded linear operator.

Definitions.

(i) The resolvent set of A is defined as

$$\rho(A) := \{\eta \in \mathbb{R} : (A - \eta I) : X \to X \text{ is bijective}\}.$$

In particular, for $\eta \in \rho(A)$, $R_A(\eta) := (A - \eta I)^{-1}$ is a bounded linear operator, referred to as the resolvent of A at η.

(ii) The spectrum of A is defined as

$$\sigma(A) := \mathbb{R} \backslash \rho(A).$$

(Here, we use $\mathbb{R}$ because our Banach space is real.)

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(iii) We denote by $\sigma_p(A)$ the point spectrum (or eigenvalues) of A, by which we mean the $\eta \in \mathbb{R}$ so that $N(A - \eta I) \neq \{0\}$. (I.e., $A - \eta I$ is not injective.)

(iv) If $\eta \in \sigma_p(A)$ and $(A - \eta I)w = 0$ for some $w \neq 0$ (i.e., $w \in N(A - \eta I) \backslash \{0\}$), then we say w is an eigenvector of η.

Theorem A.D.6. Suppose H is a real infinite-dimensional Hilbert space, and $K : H \to H$ is compact. Then:

(i) $0 \in \sigma(K)$

(ii) $\sigma(K) \backslash \{0\} = \sigma_p(K) \backslash \{0\}$

(iii) Either $\sigma(K)$ is finite or $\sigma(K) \backslash \{0\}$ is a countable sequence tending toward 0.

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Theorem 6.2.5. Suppose $U \subset \mathbb{R}^n$ is open and bounded, a^{ij}, b^i, $c \in L^\infty(U)$ $(\forall\, i, j \in \{1, 2, \ldots n\})$, and L is uniformly elliptic. Then:

(i) There exists a set $\Sigma \subset \mathbb{R}$, at most countable, so that

$$\begin{aligned} Lu &= \lambda u + f, \quad \text{in } U \\ u &= 0, \quad \text{on } \partial U, \end{aligned}$$

has a unique weak solution $u \in H_0^1(U)$ for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$.

(ii) If Σ from (i) is infinite, then its elements $\Sigma = \{\lambda_k\}_{k=1}^\infty$ can be arranged in a nondecreasing sequence so that

$$\lim_{k \to \infty} \lambda_k = +\infty.$$

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Note. Item (i) asserts that for each $\lambda \notin \Sigma$, the operator

$$L - \lambda I : H_0^1(U) \to L^2(U)$$

is a bijection, and correspondingly so is the resolvent operator

$$R_L(\lambda) := (L - \lambda I)^{-1} : L^2(U) \to H_0^1(U).$$

Proof of Theorem 6.2.5

1. Let γ be as in Theorem 6.2.2, so that

$$B[u,u] \geq \beta \|u\|^2_{H^1(U)} - \gamma \|u\|^2_{L^2(U)}, \quad \forall\, u \in H^1_0(U).$$

Notice that if γ satisfies this estimate, then any value larger than γ also works. Using this, we can take $\gamma > 0$ without loss of generality.

Recall from Theorem 6.2.3 that if $\mu = -\lambda \geq \gamma$, then the equation in Item (i) has a unique weak solution $u \in H^1_0(U)$ for each $f \in L^2(U)$. I.e., we see immediately that if $\lambda \leq -\gamma$, then $\lambda \notin \Sigma$.

We are left to consider $\lambda > -\gamma$.

Proof of Theorem 6.2.5

2. For any $\lambda > -\gamma$, we'll apply Theorem 6.2.4 (the Fredholm Alternative) to $L - \lambda I$. According to Theorem 6.2.4, the equation

$$\begin{aligned} Lu - \lambda u &= f, \quad \text{in } U \\ u &= 0, \quad \text{on } \partial U \end{aligned}$$

can be uniquely solved for all $f \in L^2(U)$ if and only if $u = 0$ is the only weak solution of

$$\begin{aligned} Lu - \lambda u &= 0, \quad \text{in } U \\ u &= 0, \quad \text{on } \partial U. \end{aligned}$$

If we add γu to both sides of this latter equation, we see that it's equivalent to

$$\begin{aligned} Lu + \gamma u &= (\gamma + \lambda)u, \quad \text{in } U \\ u &= 0, \quad \text{on } \partial U. \end{aligned} \tag{*}$$

Proof of Theorem 6.2.5

If we now let $K = \gamma L_\gamma^{-1}$ be as in the proof of Theorem 6.2.4, then we can express solutions of (*) as

$$u = L_\gamma^{-1}[(\gamma+\lambda)u] = \frac{\gamma+\lambda}{\gamma} Ku. \qquad (**)$$

More precisely, $u \in H_0^1(U)$ solves (*) if and only if it solves (**).

We see that (**) is just an eigenvalue problem for the compact operator K,

$$Ku = \frac{\gamma}{\gamma+\lambda} u.$$

This allows us to apply Theorem A.D.6. It asserts that the allowable values of the ratio $\frac{\gamma}{\gamma+\lambda}$ form a set that is at most countable and tends to 0. Recall that we're taking $\lambda > -\gamma$, so these allowable values are all positive.

Proof of Theorem 6.2.5

The allowable values of $\frac{\gamma}{\gamma+\lambda}$ can be expressed as

$$\mu_k = \frac{\gamma}{\gamma + \lambda_k},$$

and either the collection $\{\mu_k\}$ is finite or the collection is countable and

$$\lim_{k\to\infty} \mu_k = 0.$$

In the former case, the collection $\Sigma = \{\lambda_k\}$ is finite, while in the latter case it is countable with

$$\lim_{k\to\infty} \lambda_k = +\infty.$$

By a choice of ordering, we can take the set $\{\lambda_k\}$ to be nondecreasing. □

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Notes. 1. We saw in Step 2 of the proof that if $\lambda \in \Sigma$, then there exists a solution $u \in H_0^1(U)\backslash\{0\}$ to the equation

$$\begin{aligned} Lu - \lambda u &= 0, \quad \text{in } U \\ u &= 0, \quad \text{on } \partial U. \end{aligned} \tag{*}$$

I.e., $\Sigma = \sigma_p(L)$.

2. We can proceed similarly in the case that we take $H_0^1(U)$ to be a complex Hilbert space. In that case, Item (i) of Theorem 6.2.5 holds precisely as stated for some (at most) countable set $\Sigma \subset \mathbb{C}$, and Item (ii) holds with

$$\lim_{k \to \infty} |\lambda_k| = +\infty.$$

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3. Even if we take $H_0^1(U)$ to be a complex Hilbert space, the eigenvalues of L are confined to $\mathbb{R}$ in many important cases. For example, the eigenvalues associated with the Laplacian operator $L = -\Delta$ are confined to $\mathbb{R}$, and more generally this is the case for operators of the form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu,$$

with $a^{ij} = a^{ji}$ for all $i, j \in \{1, 2, \ldots, n\}$.

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Theorem 6.2.6. Suppose $U \subset \mathbb{R}^n$ is open and bounded, a^{ij}, b^i, $c \in L^\infty(U)$ ($\forall\, i, j \in \{1, 2, \dots n\}$), and L is uniformly elliptic. For Σ as in Theorem 6.2.5, if $\lambda \notin \Sigma$ then there exists a constant C so that for any $f \in L^2(U)$ and corresponding (uniquely defined) weak solution $u \in H_0^1(U)$ to

$$
\begin{aligned}
Lu &= \lambda u + f, \quad \text{in } U \\
u &= 0, \quad \text{on } \partial U,
\end{aligned}
$$

we have the inequality

$$
\|u\|_{L^2(U)} \le C \|f\|_{L^2(U)}.
$$

The constant C depends only on λ, U, and the coefficients of L.

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Note. In particular, this theorem states that for all $\lambda \notin \Sigma$, the resolvent operator

$$R_L(\lambda) := (L - \lambda I)^{-1},$$

is bounded as a map $R_L(\lambda) : L^2(U) \to L^2(U)$.

Proof of Theorem 6.2.6

Suppose not. Then for some $\lambda \notin \Sigma$ we can find sequences $\{\tilde{u}_m\}_{m=1}^{\infty} \subset H_0^1(U)$ and $\{\tilde{f}_m\}_{m=1}^{\infty} \subset L^2(U)$ so that

$$\begin{aligned} L\tilde{u}_m &= \lambda \tilde{u}_m + \tilde{f}_m, \quad \text{in } U \\ \tilde{u}_m &= 0, \quad \text{on } \partial U, \end{aligned}$$

in the weak sense, but

$$\|\tilde{u}_m\|_{L^2(U)} > m\|\tilde{f}_m\|_{L^2(U)}$$

for all $m = 1, 2, \ldots$.

Proof of Theorem 6.2.6

By a choice of scaling,

$$u_m := \frac{\tilde{u}_m}{\|\tilde{u}_m\|_{L^2(U)}}, \quad f_m = \frac{\tilde{f}_m}{\|\tilde{u}_m\|_{L^2(U)}},$$

we can take $\{u_m\}_{m=1}^{\infty}$ and $\{f_m\}_{m=1}^{\infty}$ to satisfy

$$\begin{aligned} Lu_m &= \lambda u_m + f_m, \quad \text{in } U \\ u_m &= 0, \quad \text{on } \partial U, \end{aligned}$$

in the weak sense, with $\|u_m\|_{L^2(U)} = 1$ and

$$\|u_m\|_{L^2(U)} > m\|f_m\|_{L^2(U)}$$

for all $m = 1, 2, \ldots$.

Since $\|u_m\|_{L^2(U)} = 1$ for all $m = 1, 2, \ldots$, we see that $\|f_m\|_{L^2(U)} < \frac{1}{m} \to 0$ as $m \to \infty$.

Proof of Theorem 6.2.6

Let's check that the sequence $\{u_m\}_{m=1}^{\infty}$ is bounded in $H_0^1(U)$.

If $B[u,v]$ denotes the bilinear form associated with L, then from Theorem 6.2.2 we have the usual lower bound estimate

$$B[u_m, u_m] \geq \beta \|u_m\|^2_{H^1(U)} - \gamma \|u_m\|^2_{L^2(U)}, \quad \forall\, m = 1, 2, \ldots.$$

This allows us to write

$$\begin{aligned}\beta \|u_m\|^2_{H^1(U)} - \gamma \|u_m\|^2_{L^2(U)} - \lambda \|u_m\|^2_{L^2(U)} &\leq B[u_m, u_m] - \lambda(u_m, u_m) \\ &= (f_m, u_m) \leq \|f_m\|_{L^2(U)} \|u_m\|_{L^2(U)},\end{aligned}$$

and using $\|u_m\|_{L^2(U)} = 1$,

$$\beta \|u_m\|^2_{H^1(U)} \leq (\gamma + \lambda) + \frac{1}{m}.$$

We see that $\{u_m\}_{m=1}^{\infty}$ is bounded in $H_0^1(U)$.

Proof of Theorem 6.2.6

Since $H_0^1(U)$ is reflexive, we can conclude from Theorem A.D.3 that there exists a subsequence $\{u_{m_j}\}_{j=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ so that

$$u_{m_j} \rightharpoonup u \quad \text{in } H_0^1(U).$$

Also, since $H_0^1(U) \subset\subset L^2(U)$, we can extract a further subsequence of $\{u_{m_j}\}_{j=1}^{\infty}$ (though let's continue to use $\{u_{m_j}\}_{j=1}^{\infty}$), so that

$$u_{m_j} \to u \quad \text{in } L^2(U).$$

(By uniqueness of weak limits, and the fact that strong convergence implies weak convergence, the the limits must agree.) Notice particularly that since $\|u_{m_j}\|_{L^2(U)} = 1$ for all $j \in \{1, 2, \dots\}$, we must have $\|u\|_{L^2(U)} = 1$.

For each $j \in \{1, 2, \dots\}$, u_{m_j} satisfies the weak problem

$$B[u_{m_j}, v] - \lambda(u_{m_j}, v) = (f_{m_j}, v), \quad \forall\, v \in H_0^1(U).$$

Proof of Theorem 6.2.6

For each $v \in H_0^1(U)$, the maps

$$u_{m_j} \mapsto B[u_{m_j}, v]$$
$$u_{m_j} \mapsto (u_{m_j}, v)$$

are bounded linear operators on $H_0^1(U)$, so by weak convergence we have the limits

$$\lim_{j\to\infty} B[u_{m_j}, v] = B[u, v]$$
$$\lim_{j\to\infty} (u_{m_j}, v) = (u, v).$$

In addition,

$$\lim_{j\to\infty} |(f_{m_j}, v)| \leq \lim_{j\to\infty} \|f_{m_j}\|_{L^2(U)} \|v\|_{L^2(U)} \leq \lim_{j\to\infty} \frac{\|v\|_{L^2(U)}}{m_j} = 0.$$

Proof of Theorem 6.2.6

If we take $j \to \infty$ in

$$B[u_{m_j}, v] - \lambda(u_{m_j}, v) = (f_{m_j}, v), \quad \forall v \in H_0^1(U),$$

we obtain

$$B[u, v] - \lambda(u, v) = 0 \quad \forall v \in H_0^1(U).$$

I.e., u is a weak solution of

$$\begin{aligned} Lu &= \lambda u, \quad \text{in } U \\ u &= 0, \quad \text{on } \partial U. \end{aligned}$$

Since $\lambda \notin \Sigma$, we know from Theorem 6.2.5 that u is uniquely defined, and since 0 is clearly a solution, this means $u = 0$. But this contradicts our observation that $\|u\|_{L^2(U)} = 1$, and so contradicts our original assumption that there exists a value $\lambda \notin \Sigma$ for which the claimed estimate doesn't hold. □