Second Order Elliptic PDE: Interior H^2 Regularity

MATH 612, Texas A&M University

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Overview

Our next goal is to determine conditions under which our weak solutions $u \in H_0^1(U)$ (or $u \in H^1(U)$ for inhomogeneous boundary conditions) have additional regularity.

We recall that Reg $(H^1) = 1 - \frac{n}{2}$, so for $n \geq 3$ we're starting with negative regularity. If we want classical solutions, we've got a long way to go.

Let's start with some formal considerations to get the ideas in mind. Suppose $u \in H^1(\mathbb{R})$ is a weak solution to Poisson's equation

$$-\Delta u = f, \quad \text{in } \mathbb{R}^n$$

for some $f \in L^2(\mathbb{R}^n)$. Although u is a weak solution, the strong formulation suggests that we might have $\Delta u \in L^2(\mathbb{R}^n)$. Could we use this to show that in fact $u \in H^2(\mathbb{R}^n)$?

Overview

Formally, we can compute

$$\|f\|_{L^2(\mathbb{R}^n)}^2 = \|\Delta u\|_{L^2(U)}^2 = \int_{\mathbb{R}^n} \sum_{i=1}^{n} u_{x_i x_i} \sum_{j=1}^{n} u_{x_j x_j} d\vec{x}$$

$$\overset{\text{parts}\times 2}{=} \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} u_{x_i x_j}^2 d\vec{x}.$$

We see that $u_{x_i x_j} \in L^2(\mathbb{R}^n)$ for all $i, j \in \{1, 2, \ldots, n\}$, so if $u \in H^1(\mathbb{R}^n)$ then $u \in H^2(\mathbb{R}^n)$.

Next, suppose that $f \in H^1(\mathbb{R}^n)$. Then we can push this idea further by setting $\tilde{u} = u_{x_i}$, so that

$$-\Delta \tilde{u} = f_{x_i} \in L^2(\mathbb{R}^n).$$

We can conclude as before that $\tilde{u} \in H^2(U)$, and if we do this for all $i \in \{1, 2, \ldots, n\}$, we can conclude that $u \in H^3(\mathbb{R}^n)$.

Overview

This type of argument is sometimes referred to as "bootstrapping." I.e., lifting yourself up by your own bootstraps.

To what extent can we make these ideas rigorous?

Interior H^2 Regularity

Theorem 6.3.1. Suppose $U \subset \mathbb{R}^n$ is open and bounded, $a^{ij} \in C^1(U)$, b^i, $c \in L^\infty(U)$ $(\forall\, i, j \in \{1, 2, \ldots n\})$, and L is uniformly elliptic. Also, suppose $f \in L^2(U)$, and $u \in H^1(U)$ (not necessarily $H_0^1(U)$) is a weak solution of

$$Lu = f \quad \text{in } U.$$

Then $u \in H^2_{\mathrm{loc}}(U)$, and for each $V \subset\subset U$, there exists a constant C, depending only on V, U, and the coefficients of L, so that

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

Interior H^2 Regularity

Note. Recall that to say that $u \in H^1(U)$ is a weak solution of

$$Lu = f \quad \text{in } U,$$

means that

$$B[u, v] = (f, v) \quad \forall\, v \in H_0^1(U).$$

Under the assumptions of Theorem 6.3.1, we have additionally that $u \in H^2_{\mathrm{loc}}(U)$.

For any $\phi \in C_c^\infty(U) \subset H_0^1(U)$, we can compute

$$\begin{aligned} B[u, \phi] &= \int_U \Big\{ \sum_{i,j=1}^n a^{ij} u_{x_i} \phi_{x_j} + \sum_{i=1}^n b^i u_{x_i} \phi + cu\phi \Big\} d\vec{x} \\ &\overset{\text{parts}}{=} \int_U \Big\{ - \sum_{i,j=1}^n (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^n b^i u_{x_i} + cu \Big\} \phi\, d\vec{x} \\ &= (Lu, \phi). \end{aligned}$$

Interior H^2 Regularity

On the other hand, since $C_c^\infty(U) \subset H_0^1(U)$, we have

$$B[u, \phi] = (f, \phi).$$

Equating the two inner products, we see that

$$(Lu - f, \phi) = 0$$

for all $\phi \in C_c^\infty(U)$, and we can conclude from a class lemma that $Lu = f$ for a.e. $\vec{x} \in U$.

I.e., Theorem 6.3.1 implies that our solution u solves the strong-form equation $Lu = f$ in a pointwise sense (a.e.).

Proof of Theorem 6.3.1

0. Since the proof is long, let's be clear at the outset about the strategy. Recall that $D_k^h u$ denotes the k^{th} difference quotient of size h,

$$D_k^h u(\vec{x}) = \frac{u(\vec{x} + h\hat{e}_k) - u(\vec{x})}{h}.$$

We'll show that for any $u \in H^1(U)$ as described in the theorem statement, and any fixed $V \subset\subset U$, there exists a constant $\tilde{C}$, depending only on U, V, and the coefficients of L, so that

$$\|D_k^h Du\|_{L^2(V)} \leq \tilde{C}(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$

for all $|h|$ sufficiently small.

Proof of Theorem 6.3.1

Using Theorem 5.8.3 (ii), we'll be able to conclude that there exists a constant $\tilde{\tilde{C}}$, depending only on U, V, and the coefficients of L, so that

$$\|D^2 u\|_{L^2(V)} \leq \tilde{\tilde{C}}(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}),$$

giving the estimate we're looking for on the second derivative. (First derivative estimates are easier.) The new constant $\tilde{\tilde{C}}$ arises because we've combined estimates in the latter inequality for all second order derivatives.

Here, recall that

$$|D^2 u| := (\sum_{|\alpha|=2} |D^\alpha u|^2)^{1/2}.$$

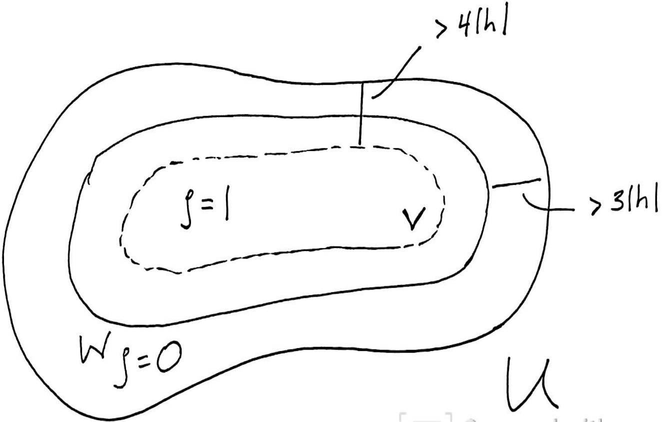
Proof of Theorem 6.3.1

1. Fix $V \subset\subset U$ and take an open set W so that $V \subset\subset W \subset\subset U$. Let $\zeta \in C_c^\infty(\mathbb{R}^n)$ denote a cut-off function so that

$$\zeta = \begin{cases} 1 & \vec{x} \in V \\ \in [0,1] & \vec{x} \in W \backslash V \\ 0 & \vec{x} \in \mathbb{R}^n \backslash W. \end{cases}$$

See figure on the next slide.

Proof of Theorem 6.3.1



Proof of Theorem 6.3.1

2. Since $u \in H^1(U)$ is a weak solution of $Lu = f$, we have

$$\int_U \Big\{ \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \Big\} d\vec{x} = \int_U f v d\vec{x}$$

for all $v \in H_0^1(U)$. We can rearrange this relation as

$$\int_U \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} d\vec{x} = \int_U \Big\{ f - \sum_{i=1}^n b^i u_{x_i} - cu \Big\} v d\vec{x}.$$

Notice that since $u \in H^1(U)$ and $f \in L^2(U)$, the function

$$\tilde{f} := f - \sum_{i=1}^n b^i u_{x_i} - cu$$

satisfies $\tilde{f} \in L^2(U)$.

Proof of Theorem 6.3.1

I.e., for $\tilde{f} \in L^2(U)$,

$$\int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} d\vec{x} = (\tilde{f}, v)$$

for all $v \in H_0^1(U)$.

The key point in the proof consists of choosing v appropriately. Thinking formally, we would like to choose $v = u_{x_k x_k}$, integrate by parts similarly as in the overview and then finish off the proof by using uniform ellipticity.

Proof of Theorem 6.3.1

3. As specified on our figure, we let

$$0 < |h| < \frac{1}{4}\mathrm{dist}(V, \partial U)$$
$$0 < |h| < \frac{1}{3}\mathrm{dist}(W, \partial U).$$

Here, we're saving some room to insert another set with a factor of $\frac{1}{2}$.

For $k \in \{1, 2, \ldots, n\}$, we set

$$v(\vec{x}) := -D_k^{-h}(\zeta(\vec{x})^2 D_k^h u(\vec{x})).$$

Notice that since $\mathrm{spt}(\zeta) \subset \overline{W}$, $|h|$ is small enough so that $v(\vec{x})$ is 0 for $\vec{x}$ close enough to ∂U. Otherwise, for each fixed h, with $|h| > 0$, $v(\vec{x})$ is a linear combination of $H^1(U)$ functions, so $v \in H_0^1(U)$.

Also, aside from $\zeta(\vec{x})^2$, which is identially 1 on V, this is a standard second order finite difference approximation of $-u_{x_k x_k}$.

Proof of Theorem 6.3.1

We now substitute v into

$$\int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} d\vec{x} = (\tilde{f}, v).$$

In the next steps of the proof, we'll denote the resulting left-hand side by A and the resulting right-hand side by B.

4. In this step, we'll develop our estimate on A. First, fix any $\tilde{U} \subset\subset U$ and $0 < |h| < \mathrm{dist}(\tilde{U}, \partial U)$.

Claim 1. We can integrate by parts in the following way,

$$\int_U u(\vec{x}) D_k^{-h} \phi(\vec{x}) d\vec{x} = -\int_U \phi(\vec{x}) D_k^h u(\vec{x}) d\vec{x}$$

for all $\phi \in C_c^\infty(\tilde{U})$. Moreover, this is true for $\phi(\vec{x}) = \zeta(\vec{x})^2 D_k^h u(\vec{x})$.

Proof of Theorem 6.3.1

To see this, we write

$$\int_U u(\vec{x}) D_k^{-h}\phi(\vec{x}) d\vec{x} = \int_U u(\vec{x}) \frac{\phi(\vec{x} - h\hat{e}_k) - \phi(\vec{x})}{-h} d\vec{x}$$

$$= - \int_{\mathrm{spt}(\phi(\cdot - h\hat{e}_k))} u(\vec{x}) \frac{\phi(\vec{x} - h\hat{e}_k)}{h} d\vec{x} + \int_{\tilde{U}} \frac{u(\vec{x})\phi(\vec{x})}{h} d\vec{x}.$$

Set $\vec{y} = \vec{x} - h\hat{e}_k$ in the first integral to get

$$- \int_{\tilde{U}} u(\vec{y} + h\hat{e}_k) \frac{\phi(\vec{y})}{h} d\vec{y} + \int_{\tilde{U}} \frac{u(\vec{x})\phi(\vec{x})}{h} d\vec{x}$$

$$= - \int_{\tilde{U}} \phi(\vec{x}) D_k^h u(\vec{x}) d\vec{x} = - \int_U \phi(\vec{x}) D_k^h u(\vec{x}) d\vec{x}.$$

Precisely the same calculation works with $\phi(\vec{x}) = \zeta(\vec{x})^2 D_k^h u(\vec{x})$, with $W \subset\subset \tilde{U} \subset\subset U$.

Proof of Theorem 6.3.1

Claim 2. We have the product rule

$$D_k^h(vw) = v^h D_k^h w + w D_k^h v,$$

for any u and v defined a.e. where evaluated. Here, $v^h(\vec{x}) := v(\vec{x} + h\hat{e}_k)$.

For this one, we compute

$$\begin{aligned}
D_k^h(vw) &= \frac{v(\vec{x} + h\hat{e}_k)w(\vec{x} + h\hat{e}_k) - v(\vec{x})w(\vec{x})}{h} \\
&= \frac{v(\vec{x} + h\hat{e}_k)w(\vec{x} + h\hat{e}_k) - v(\vec{x} + h\hat{e}_k)w(\vec{x})}{h} \\
&+ \frac{v(\vec{x} + h\hat{e}_k)w(\vec{x}) - v(\vec{x})w(\vec{x})}{h} \\
&= v^h(\vec{x}) D_k^h w(\vec{x}) + w(\vec{x}) D_k^h v(\vec{x}).
\end{aligned}$$

Proof of Theorem 6.3.1

Now, for $v = -D_k^{-h}(\zeta^2 D_k^h u)$, we use Claims 1 and 2 to compute

$$
\begin{aligned}
A &= \sum_{i,j=1}^{n} \int_U a^{ij} u_{x_i} v_{x_j} d\vec{x} = -\sum_{i,j=1}^{n} \int_U a^{ij} u_{x_i} \left\{ D_k^{-h}(\zeta^2 D_k^h u) \right\}_{x_j} d\vec{x} \\
&\overset{\text{claim1}}{=} \sum_{i,j=1}^{n} \int_U D_k^h(a^{ij} u_{x_i})(\zeta^2 D_k^h u)_{x_j} d\vec{x} \\
&\overset{\text{claim2}}{=} \sum_{i,j=1}^{n} \int_U \left\{ a^{ij,h}(D_k^h u_{x_i}) + u_{x_i}(D_k^h a^{ij}) \right\} (\zeta^2 D_k^h u_{x_j} + 2\zeta\zeta_{x_j} D_k^h u) d\vec{x} \\
&= \sum_{i,j=1}^{n} \int_U a^{ij,h}(D_k^h u_{x_i})(\zeta^2 D_k^h u_{x_j}) d\vec{x} \\
&+ \sum_{i,j=1}^{n} \int_U \left\{ a^{ij,h}(D_k^h u_{x_i})(2\zeta\zeta_{x_j} D_k^h u) + u_{x_i}(D_k^h a^{ij})(\zeta^2 D_k^h u_{x_j}) \right. \\
&+ \qquad \left. u_{x_i}(D_k^h a^{ij}) 2\zeta\zeta_{x_j}(D_k^h u) \right\} d\vec{x}.
\end{aligned}
$$

Proof of Theorem 6.3.1

In this last expression, we'll set

$$A_1 = \sum_{i,j=1}^{n} \int_U a^{ij,h}(D_k^h u_{x_i})(\zeta^2 D_k^h u_{x_j}) d\vec{x},$$

and we'll denote the remaining terms A_2.

By uniform ellipticity, we see that

$$A_1 \geq \theta \int_U \zeta^2 |D_k^h Du|^2 d\vec{x},$$

for some $\theta > 0$.

Proof of Theorem 6.3.1

For A_2, we can write

$$\begin{aligned} |A_2| \leq \sum_{i,j=1}^{n} \int_U \Big\{ & |a^{ij,\,h}||D_k^h u_{x_i}|2\zeta|\zeta_{x_j}||D_k^h u| + |u_{x_i}||D_k^h a^{ij}|\zeta^2|D_k^h u_{x_j}| \\ + \quad & |u_{x_i}||D_k^h a^{ij}|2\zeta|\zeta_{x_j}||D_k^h u| \Big\} d\vec{x}. \\ \leq C_1 \int_U & \Big\{ \zeta|D_k^h Du||D_k^h u| + \zeta|Du||D_k^h Du| + \zeta|Du||D_k^h u| \Big\} d\vec{x} \end{aligned}$$

for some constant C_1. Here, the terms shaded red are bounded and incorporated into the constant C_1.

We now want to estimate this final expression in such a way that the contribution from the terms in blue is small. We'll use the ϵ-Young's inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2.$$

Proof of Theorem 6.3.1

We get

$$(\zeta|D_k^h Du|)|D_k^h u| \leq \epsilon\zeta^2|D_k^h Du|^2 + \frac{1}{4\epsilon}|D_k^h u|^2$$
$$(\zeta|D_k^h Du|)|Du| \leq \epsilon\zeta^2|D_k^h Du|^2 + \frac{1}{4\epsilon}|Du|^2.$$

We'll also use the usual Young's inequality to write

$$\zeta|Du||D_k^h u| \leq \zeta\left(\frac{1}{2}|Du|^2 + \frac{1}{2}|D_k^h u|^2\right).$$

Combining these observations, we obtain the inequality

$$|A_2| \leq 2C_1\epsilon \int_U \zeta^2|D_k^h Du|^2 d\vec{x} + K_\epsilon \int_W |D_k^h u|^2 + |Du|^2 d\vec{x},$$

for some constant K_ϵ that grows as ϵ is reduced.

Proof of Theorem 6.3.1

In both integrals, the integration is only over W due to the support of ζ. Since ζ has been incorporated into the constant K_ϵ for the second integral, this has been made explicit.

We now choose $\epsilon > 0$ small enough so that $2C_1\epsilon \leq \frac{\theta}{2}$. Then

$$|A_2| \leq \frac{\theta}{2}\int_U \zeta^2 |D_k^h Du|^2 d\vec{x} + K_\epsilon \int_W |D_k^h u|^2 + |Du|^2 d\vec{x}.$$

From Theorem 5.8.3 (i), we know that since $u \in H^1(U)$ and $W \subset\subset U$, there exists a constant C_2, depending only on W and U, so that

$$\|D_k^h u\|_{L^2(W)} \leq C_2 \|Du\|_{L^2(U)}$$

for all $0 < |h| < \frac{1}{3}\mathrm{dist}(W, \partial U)$. (In the statement of Theorem 5.8.3, $\frac{1}{2}$ is used in place of $\frac{1}{3}$.)

Proof of Theorem 6.3.1

This allows us to write

$$|A_2| \leq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 d\vec{x} + \tilde{K}_\epsilon \int_U |Du|^2 d\vec{x}.$$

Combining these observations, we see that

$$\begin{aligned} A &= A_1 + A_2 \geq A_1 - |A_2| \\ &\geq \theta \int_U \zeta^2 |D_k^h Du|^2 d\vec{x} - \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 d\vec{x} - \tilde{K}_\epsilon \int_U |Du|^2 d\vec{x} \\ &= \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 d\vec{x} - \tilde{K}_\epsilon \int_U |Du|^2 d\vec{x}. \end{aligned}$$

Proof of Theorem 6.3.1

5. In this step, we'll develop our estimate on $B = (\tilde{f}, v)$, recalling that $\tilde{f} = f - \sum_{i=1}^{n} b^i u_{x_i} - cu$ and $v(\vec{x}) := -D_k^{-h}(\zeta(\vec{x})^2 D_k^h u(\vec{x}))$.

We have

$$|B| = |(\tilde{f}, v)| = |(f - \sum_{i=1}^{n} b^i u_{x_i} - cu, v)|$$

$$\leq C_3 \int_U \Big(|f| + |Du| + |u|\Big)|v| d\vec{x}$$

$$\stackrel{\epsilon-\text{Young's}}{\leq} 3C_3\epsilon \int_U |v|^2 d\vec{x} + \mathcal{K}_\epsilon \int_U |f|^2 + |Du|^2 + |u|^2 d\vec{x}.$$

Here,

$$\int_U |v|^2 d\vec{x} = \int_U |D_k^{-h}(\zeta^2 D_k^h u)|^2 d\vec{x}.$$

Proof of Theorem 6.3.1

Similarly as with our discussion for v, we have $\zeta^2 D_k^h u \in H_0^1(U)$. Also, since $\operatorname{spt}(\zeta) \subset \overline{W}$, we can write

$$\int_U |D_k^{-h}(\zeta^2 D_k^h u)|^2 d\vec{x} = \int_{W_h} |D_k^{-h}(\zeta^2 D_k^h u)|^2 d\vec{x},$$

for a set W_h so that

$$0 < |h| < \frac{1}{2}\operatorname{dist}(W_h, \partial U).$$

(Using W_h because of the shift on ζ.)

We can now apply Theorem 5.8.3 (i) to see that

$$\begin{aligned}\int_U |v|^2 d\vec{x} &= \int_{W_h} |D_k^{-h}(\zeta^2 D_k^h u)|^2 d\vec{x} \\ &\leq C_4 \int_U |D(\zeta^2 D_k^h u)|^2 d\vec{x}.\end{aligned}$$

Proof of Theorem 6.3.1

Using $\mathrm{spt}(\zeta) \subset \overline{W}$ and the product rule, we can write

$$\begin{aligned} C_4 \int_U |D(\zeta^2 D_k^h u)|^2 d\vec{x} &= C_4 \int_W |(D\zeta^2) D_k^h u + \zeta^2 D D_k^h u|^2 d\vec{x} \\ &\leq C_5 \int_W |D_k^h u|^2 + \zeta^2 |D_k^h D u|^2 d\vec{x} \\ &\leq C_6 \int_U |Du|^2 + \zeta^2 |D_k^h D u|^2 d\vec{x}. \end{aligned}$$

In obtaining the second summand in the second line, a factor of ζ^2 was incorporated into C_5, and we observed that D and D_k^h commute. In obtaining the first summand in the third line, we used Theorem 5.8.3 (i).

Proof of Theorem 6.3.1

To recap, we now have

$$|B| \leq 3C_3\epsilon \int_U |v|^2 d\vec{x} + \mathcal{K}_\epsilon \int_U |f|^2 + |Du|^2 + |u|^2 d\vec{x},$$

and

$$\int_U |v|^2 d\vec{x} \leq C_6 \int_U |Du|^2 + \zeta^2 |D_k^h Du|^2 d\vec{x}.$$

Combining, we see that for some constants C_7 and $\tilde{\mathcal{K}}_\epsilon$, we can write

$$|B| \leq C_7\epsilon \int_U \zeta^2 |D_k^h Du|^2 d\vec{x} + \tilde{\mathcal{K}}_\epsilon \int_U |f|^2 + |Du|^2 + |u|^2 d\vec{x}.$$

We choose ϵ small enough so that $C_7\epsilon \leq \frac{\theta}{4}$, giving

$$|B| \leq \frac{\theta}{4} \int_U \zeta^2 |D_k^h Du|^2 d\vec{x} + \tilde{\mathcal{K}}_\epsilon \int_U |f|^2 + |Du|^2 + |u|^2 d\vec{x}.$$

Proof of Theorem 6.3.1

6. Recalling that $A = B$ and

$$A \geq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 d\vec{x} - \tilde{K}_\epsilon \int_U |Du|^2 d\vec{x}$$

we now have

$$\begin{aligned} 0 = A - B &\geq A - |B| \\ &\geq \frac{\theta}{4} \int_U \zeta^2 |D_k^h Du|^2 d\vec{x} - \mathbf{K}_\epsilon \int_U |f|^2 + |Du|^2 + |u|^2 d\vec{x}. \end{aligned}$$

Turning this around, we see that

$$\int_U \zeta^2 |D_k^h Du|^2 d\vec{x} \leq C_8 \int_U |f|^2 + |Du|^2 + |u|^2 d\vec{x}.$$

Since $\zeta \equiv 1$ on V, we have

$$\int_V |D_k^h Du|^2 d\vec{x} \leq C_8 \int_U |f|^2 + |Du|^2 + |u|^2 d\vec{x}.$$

Proof of Theorem 6.3.1

This is true for all $0 < |h| < \frac{1}{4}\mathrm{dist}(V, \partial U)$. Noting that we have an estimate of this form for each k, we can conclude from (a slight restatement of) Theorem 5.8.3 (ii) that $u_{x_j} \in H^1(V)$ for each $j \in \{1, 2, \ldots, n\}$, and

$$\|D^2 u\|_{L^2(V)}^2 \leq C_9 \int_U |f|^2 + |Du|^2 + |u|^2 d\vec{x}.$$

Since $V \subset\subset U$ is arbitrary, we can conclude that $u \in H^2_{\mathrm{loc}}(U)$ and

$$\|u\|_{H^2(V)} \leq C_{10}(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}).$$

7. Last, we need to show that on the right-hand side of this final inequality, we can replace $\|u\|_{H^1(U)}$ with $\|u\|_{L^2(U)}$.

Proof of Theorem 6.3.1

Recalling that $V \subset\subset W \subset\subset U$, we introduce a new open set $\tilde{V}$ so that $V \subset\subset \tilde{V} \subset\subset W$. Proceeding exactly as in Steps 1-6 with V, $\tilde{V}$, W respectively replacing the roles of V, W, U, we immediately arrive at the inequality

$$\|u\|_{H^2(V)} \leq C_{11}(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}).$$

Finally, we introduce one last open set $\tilde{W}$ so that $W \subset\subset \tilde{W} \subset\subset U$, and we take $\zeta \in C_c^\infty(\mathbb{R}^n)$ to be a new cut-off function satisfying

$$\zeta(\vec{x}) = \begin{cases} 1 & \vec{x} \in W \\ \in [0,1] & \vec{x} \in \tilde{W} \setminus W \\ 0 & \vec{x} \in \mathbb{R}^n \setminus \tilde{W}. \end{cases}$$

Proof of Theorem 6.3.1

We now return in the proof all the way to the end of Step 3, when we wrote the relation

$$\sum_{i,j=1}^{n} \int_U a^{ij} u_{x_i} v_{x_j} d\vec{x} = (\tilde{f}, v) \quad \forall\, v \in H_0^1(U).$$

We'll check in the homework that if we set $v = \zeta^2 u$ and proceed similarly as above, we obtain the inequality

$$\int_W |Du|^2 d\vec{x} \leq C_{12}(\|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2),$$

so that

$$\|u\|_{H^1(W)} \leq C_{13}(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}),$$

and this gives the inequality stated in the theorem. □